Gauge Theory of Gravity: Foundations, the charge concept, and a numeric solution

Eichtheorie der Gravitation: Grundlagen, das Konzept der Ladung und eine nummerische Lösung

Diploma thesis
by

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Zusammenfassung

Das urpsrüngliche Thema dieser Diplomarbeit war die Frage: *Gibt es Torsionsmonopole in der Gravitation in Analogie zu dem Diracschen Monopol im Elektromagnetismus?* Obwohl schnell eine positive Antwort gefunden war, führte der Wunsch nach einem tieferen Verständnis zu einer allgemeineren Arbeit: Der Diracsche Monopol lässt sich am besten in einer eichtheoretischen Formulierung des Elektromagnetismus verstehen, im Besonderen als topologischer Effekt auf einem $U(1)$-Bündel. Um also Analogien zu finden, sollte Gravitation ebenso als Eichtheorie formuliert werden. Dies wird das Thema des ersten Kapitels sein: Nach einem philosophischen Vorwort (das lediglich eine persöhnliche Sichtweise darstellt) folgt eine Einführung in allgemeine Eichtheorien und es wird gezeigt wie eine Eichtheorie der affinen Gruppe mit einer Theorie der Gravitation identifiziert werden kann. Dieses Thema ist nicht neu und wurde in der Literatur bereits ausführlich diskutiert. Jedoch wird hier eine andere Darstellung präsentiert, um auf andere Motivationen einzugehen. Besonders die Beschreibung der Symmetriebrechung in Abschnitt 1.4 und die Dimensionsanalyse von Eichtheorien in Abschnitt 1.6 sind eigene Beiträge.

Nach dieser Vorbereitung werden im zweiten Kapitel verschiedene Ladungsbegriffe in Eichtheorien definiert, einschließlich der magnetischen Ladung in Analogie zum Diracschen Monopol ein. Die vorangegangene Dimensionsanalyse hilft, die Beziehungen zwischen den verschiedenen Arten der Ladungen zu verstehen. Anschließend werden verschiedene Lösungen von Eichtheorien der Gravitation nach solchen Ladungen untersucht. Das wichtigste Ergebnis wird folgendes sein: Der Schwarzschildsche Massenparameter ist eine quasi-elektrische Monopolladung der Zeittranslation während der NUT-Parameter eine quasi-magnetische Monopolladung der Zeittranslation ist.

Das dritte Kapitel hat das zusätzliche Problem zum Gegenstand, *eine Lösung der Einstein-Proca-Gleichungen zu finden.* Die Motivation zu diesem Problem ist eine Arbeit von Obukhov et al. [27], in der die Äquivalenz der metrisch-affinen Eichtheorie und der Einstein-Proca-Theorie für einen Spezialfall gezeigt wurde. Auch die Diskussion der Symmetriebrechung, insbesondere die Betrachtung der Spaltung der Linearen Gruppe, deutet

diese Äquivalenz an. Wir geben die Lagrangedichte der Einstein-Proca-Theorie an, leiten die Feldgleichungen ab und integrieren sie nummerisch. Dabei legen wir besonderen Wert auf die Diskussion der Integrationskonstanten und präsentieren eine Reihenentwicklung der Lösung um den Ursprung. Die nummerische Lösung wird anhand vieler Diagramme dargestellt und diskutiert. Wir versuchen auch, die Stabilität der Lösung zu untersuchen, wobei wir uns nach dem Verfahren von Jetzer [19] richten.

Schließlich sollte man erwähnen, dass ein großer Teil dieser Arbeit darin bestand, entsprechende Routinen für verschiedene Computeralgebra-Systeme zu entwickeln. So zum Beispiel die Bibliothek `magtools` für Reduce, die die wichtigsten Routinen im Umfeld der metrisch-affinen Eichtheorie implementiert. In Kapitel 3 wird Reduce benutzt, um die Feldgleichungen explizit zu berechnen. Diese Ergebnisse werden dann exportiert, um die Gleichungen mit Maple zu vereinfachen und zu integrieren. Anhang D zeigt einige Computeralgebra-Dateien.

Die Abschnitte 1.7, 2.4 und 3.5 sind Zusammenfassungen der jeweiligen Kapitel. Der Anhang enthält zudem eine knappe Erläuterung der Konventionen (inklusive der des Äußeren Kalküls) und eine Zusammenfassung einiger mathematischer Grundlagen.

Overview

This diploma thesis was originally supposed to answer the question: *Are there torsion monopoles in gravity in analogy to the Dirac monopole in electromagnetism?* It did not take long to find a positive answer. However, the urge for a deeper understanding led to more general work: The Dirac monopole may most generally be understood in a gauge theoretical formulation of electromagnetism, specifically as a topological effect in the respective $U(1)$-bundle. Thus, in order to draw analogies, gravity should also be formulated as a gauge theory. This is what we will present in chapter 1: After a philosophical preface (that merely represents my personal point of view) we will introduce gauge theories in general and show how a gauge theory of the affine group can be identified with gravity. This subject is not new and was already extensively discussed in the literature. However, in order to build on different motivations we present a different formulation here. Especially the symmetry breaking mechanism described in section 1.4 and the dimensional analysis of gauge theories in section 1.6 are my own contributions.

After these preparations, in chapter 2, we will define the notion of different charges in gauge theories – including, of course, magnetic charges in analogy to the Dirac monopole. The previous dimensional analysis helps us to understand the relations between different kinds of charges to be defined. We then analyze different solutions of gauge theories of gravity for such charges and find (*quasi*-)magnetic charges in the torsion field, indeed. The most important result will be: The Schwarzschild mass parameter is a quasi-electric monopole charge of the time translation whereas the Taub-NUT parameter is a quasi-magnetic monopole charge of the time translation.

The subject of chapter 3 is an additional problem to be discussed in this diploma thesis, namely *to find a solution of the Einstein-Proca system of partial differential equations.* The motivation for this problem is the work by Obukhov et al. [27] who found that a special case of the metric-affine gauge theory of gravity is effectively equivalent to the coupled Einstein-Proca theory. Also our discussion of symmetry breaking, in particular the investigation of the splitting of the linear group, insinuates this equivalence in a very intuitive way. We set up the lagrangian of the Einstein-Proca theory, derive the field equations, and integrate them numerically. We especially address the problem of integration constants and also present a power series expansion of the solution at the origin. Of course, the main result of this chapter will be the numeric solution which has to be discussed by means of numerous graphs. Interesting is, e.g., that it has no horizon. We will also try to discuss the stability of our solution following the scheme of Jetzer [19].

Finally one should know that a major part of this work was to develop the appropriate computer algebra routines. This is, e.g., the library `magtools` for Reduce that implements the most important routines in the context of metric-affine gauge theory of gravity. In chapter 3 we used Reduce to explicitely calculate the field equations and exported them to Maple in order to perform simplifications and the numerical integration. In appendix D we display some computer algebra files.

For a summary of the results of each chapter see sections 1.7, 2.4, and 3.5 at the end of each chapter. Finally, the appendix includes a brief summary of our conventions (including those of exterior algebra) and a mathematical reference.

Contents

Chapter 1

Developing a gauge theory of gravity

1.1 Introduction

Newtonian physics was dominated by the idea of forces acting on particles. It was a phenomenological problem to find all possible forces, how they act on particles, what particles there are, and how they differ. The idea of gauge theories offers a very beautiful and powerful explanation of the existence of forces and particles: It seems that everything is merely a reflection of a symmetry – particles as well as forces. Given a symmetry – whatever the origin is – invariants exist and form currents that interact. Particles are our impression of invariants (that are in some cases quantized) and forces are the result of the general relation (connection) of the symmetry at one point in spacetime and a neighboring point. The first gauge theory was electromagnetism. Although one did not know about the $U(1)$-gauge structure of this theory, its covariant formulation, found at the beginning and in the middle of this century, clearly displays the basic structure of a gauge theory: The *field strength* describes all forces, and the (minimal) coupling of forces to matter is determined by a lagrangian in which the *potential* acts on (is multiplied to) the matter field. In the fifties, Yang and Mills [34] for the first time formulated the $SU(2)$-gauge theory by strictly keeping to the electromagnetic paradigm. They pointed out that such a theory is related to the conservation of charged currents. At about the same time, Utiyama [33] formulated the general gauge theory of a semi-simple Lie group. These theories, as they explain the electro-weak and strong *forces*, were supplemented by the great success of *particle* physics to classify all leptons as representations of the electro-weak symmetry and all hadrons as representations of the flavor symmetry. O'Raifeartaigh [6] gives more detailed insight into the history of gauge theories.

This work will mainly be concerned with a theory of gravity – one of the four fundamental forces which should also be representable in the framework of gauge theory. However, the obvious difference between the *external* spacetime symmetries and *internal* symmetries (as considered by Yang and Mills) causes some difficulties for a uniform formulation of all forces. Some ad-hoc assumption (the *soldering*) solves basic problems but perhaps assaults the beauty of the theory. Hehl et al. [17] formulated such a *metric-affine gauge theory of gravity* (MAG). It is the purpose of this chapter to give an introduction to gauge theories in general and to MAG specifically. The introduction differs from others in some aspects (as, e.g., the discussion of the breaking mechanism in gauge theories of gravity) and also includes a dimensional analysis. As a preparation for the following chapter we learn about the teleparallel theory and the Kaluza-Klein formulation of the coupled Einstein-Maxwell theory.

1.2 Philosophical preface

This section tries to give a deeper motivation for the subsequent introduction of a gauge theory of gravity. It is especially addressed to the problem of *objects* that a theory involves axiomatically: May a theory postulate only such objects that are a *natural* realization of notions that physicists have developed historically (such as currents), or may a theory postulate any objects (such as strings) as long as it also gives rules how to interpret the results and compare them with nature, or are there some philosophical arguments in favor of a mechanism to constitute and postulate objects? The ideas are inspired by a short discussion with Peter Mittelstaedt, his work [22] and [23], and his citation of David Hume and Hermann Weyl therein.

[*1.1*] In 1739, David Hume stated a quite deep and final idea in his *Treatise of Human Nature*: All there exists from the point of view of *one* individual are thoughts and observations[1]. (Since we don't want to talk about thoughts here we will restrict this to observations.) All one can observe are qualities that, perhaps, one conceives as properties of objects – one may never observe objects themselves. Hence, the idea that objects really exist is nothing but imagination. All the notions mankind developed to denote objects thus have no eligibility a-priori.

[*1.2*] *Theories* are methods that map observations into other observations in such a way that these mappings represent laws the observer experiences in nature. This definition is consistent with Hume's point of view because it does not presume that theories are

[1] As long as the existence of other individuals/observers has not been postulated, there is no inter-subjectivity and this idea seems final, indeed. We postulate other observers later, in paragraph [*1.3*].

concerned with objects. In order to succeed, theories may use any kind of hypothetical model to derive these mappings. These models may in turn postulate any hypothetical *something* – which, however, may not be understood as an object a-priori, even if the theory is successful. The excellence of a theory is determined *only* by the variety of laws (the theory represents) which are not refuted but refutable. This means that a theory which is based on purely abstract and hypothetical notions has no justification until is gives rules how to relate the results to nature and until is reproduces observable laws. Theories that are based on historical notions of objects usually have such a justification from the very beginning. But this argument does not mean in principle that the former type of theory is *less excellent* than the latter.

[*1.3*] The crucial point to *constitute the notion of an object* is to postulate that there exist more observers than one individual and that there is a consensus between these observers concerning particular properties they observe. Such properties are called *objective*. At about 1930, Hermann Weyl[2] proposed that we may identify an object by the set of invariant properties which are observed by all observers, i.e. the set of objective properties. Thereby we define the notion of an object. If a theory is to be *objective*, it should include the notion of such objects. The theory needs to postulate first what the set of all observers is, then it searches for possible invariant properties and defines the set of objects as the set of all possible invariant property tuples.

[*1.4*] In *gauge theories* the set of observers is replaced by the symmetry group. (The set of observers now being an orbit of this group.) In this sense, postulating the symmetry means postulating all possible observers. Invariants are charges of the Casimir operators of this group (see section 2.1.3) and objects are elements in representation spaces of the group classified by their Casimir charges. Thus, the philosophical considerations above are in perfect harmony with particle physics. In this spirit, a gauge theory should postulate the symmetry first. The notion of objects is to be derived later.

The preceding argumentation neglected the historical development of theories and is of theoretical value rather than of a practical one. A reflection in paragraph [*1.3*] shows that this approach to the constitution of objects is not very constructive and hence does not describe the historical development. The crucial point is that different observers, in order to arrive at a consensus on objective observations, need some language to communicate. And this language must be appropriate to recognize invariant properties. Such a language is naturally given if we *postulate* measuring devices. In the following introduction of gauge theories, though, we will simply introduce a space V of properties with a comparison operator $-$ (*minus*) defined on it. This is an appropriate mathematical language to compare

[2] *Philosophie der Mathematik und Naturwissenschaft.* R. Oldenbourg, München, 3rd edition, (1996).

observations. In history, however, earlier theories were necessary to developed notions (like *charge* or *current*) and thereby a language without which a recognition of gauge symmetries would not have been possible. Hence, to reflect the historical development, a gauge theory may start to postulate invariants in a conventional language and then conclude symmetries instead of postulating a symmetry first in order to define what an invariant is.

1.3 Gauge Theories

There is one main difference between the following introduction of gauge theories and those presented in standard textbooks about quantum field theory: We introduce the notion of a connection *before* we require local gauge invariance. We want to stress the meaning of the connection as a comparison operator and not as some field we only introduce in order to rescue the postulate of local gauge invariance.

[*1.5*] We have an n-dimensional manifold M called spacetime in which, at each point, we may observe a property $\psi \in V$ in the space V of properties if we are (or our measuring device is) presently located at this point.

[*1.6*] *There exists a semi-simple symmetry Lie group* G. This implies that there must exist a representation of G on the space V of properties.

[*1.7*] Suppose that, as we walk around (or displace our measuring device ...) in spacetime, we realize that our observations depend on the path we walked. In particular, if we walk around a loop γ, the property ψ_a observed afterwards might differ from the property ψ_b observed before – although we observe them at the same point. Obviously, we have (or our measuring device has) been influenced by this walk around the loop. (We became another observer in the set of observers.) This influence must be G-valued and we may write the influence functionally as $\psi_a = g \cdot \psi_b$, where $g \in G$. In general, we should consider such an influence for any walk $\gamma : [0, l] \to M$. We do not consider this influence g as arbitrary; there must exist some special kind of correlation: *There exists a functional dependence* $g = g(\gamma)$ *between the influence* g *on the observer and the displacement path* γ.

[*1.8*] We develop a model: We introduce a *local* field $\psi : U \to V$, where U is a small neighborhood of our present location p. We think of ψ as assigning properties to any point $q \in U$ as we would observe them if we *were* present at q. However, we cannot observe $\psi(q)$ since we would have to walk up to q and thereby influence our observation. Instead, if we walked along γ from p to q, then we would observe the property $g(\gamma) \cdot \psi(q)$, cf. figure 1.1. Having postulated $\psi : U \to V$, we can write down the following difference that describes

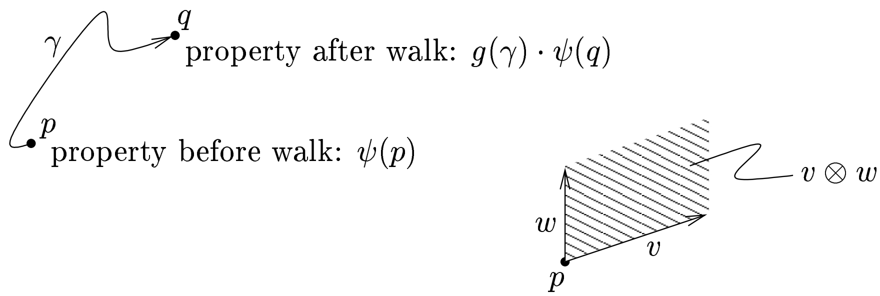


Figure 1.1: The left diagram illustrates the influence $g(\gamma)$ of a displacement γ on the local property field ψ. The right diagram represents an infinitesimal 2-plane attached at some point p. The influence $g\big(\partial(v\otimes w)\big)$ of a displacement around this 2-plane is the component $v\rfloor w\rfloor F$ of the field strength.

how this observation changes as we walk along a path $\gamma:[0,l]\to M$ with $\gamma(0)=p$:

$$g(\gamma)\cdot\psi(\gamma(l))-\psi(p)\ . \tag{1.1}$$

From our point of view this difference is *reality*, whereas the ordinary difference $\psi(q)-\psi(p)$ is not observable. If $g(\gamma)$ is given for all paths γ, we may *connect*, i.e. compare, observed properties at two points by considering the difference (1.1). This comparison has an observational, i.e. physical meaning, whereas $\psi(q)-\psi(p)$ has not. In other words: *If g is given for any path γ, we may define the comparison (1.1) which has physical meaning.*

[*1.9*] We consider the limit of this difference for short paths. With $u=\partial_s\gamma(s)|_{s=0}\in T_pM$ we get:

$$\begin{aligned}&\lim_{l\to 0}\frac{1}{l}\big[g(\gamma)\cdot\psi(\gamma(l))-\psi(p)\big]\\&=\lim_{l\to 0}\frac{1}{l}\Big[\big(\mathrm{id}+l\,A(p,u)\big)\cdot\psi(p+lu)-\psi(p)\Big]\\&=u\rfloor d\psi+A(p,u)\cdot\psi(p)\ .\end{aligned} \tag{1.2}$$

In the second line we used the first order expansion of $\gamma(l)=p+l\,u$ and $g(\gamma)=\mathrm{id}+l\,A(p,u)$. (The latter stems from the exponential map on a semi-simple Lie group. This is the reason why we took G to be semi-simple.) Hence, $A(p,u)$ is an element of the Lie-algebra $\mathcal{G}$. Now we impose A to be linear in u and write $u\rfloor A(p)$ instead of $A(p,u)$, with the $\mathcal{G}$-valued 1-form $A(p)\in\Lambda^1(U,\mathcal{G})$. Then this limit defines a differentiation that we call *covariant derivative*

$$u\rfloor D\psi(p):=u\rfloor\big[d\psi+A(p)\dot{\wedge}\psi(p)\big]\ . \tag{1.3}$$

The operation $\dot{\wedge}$ denotes an exterior product together with an application of the $\mathcal{G}$-value of A on ψ. (For all conventions see also appendix A.) The quantity A is called *linear* *connection* because it *connects* (allows to compare) observations at neighboring points in U. The $\mathcal{G}$-valued 2-form F with

$$F \dot{\wedge} \psi := DD\psi = dd\psi + A \dot{\wedge} d\psi + d(A \dot{\wedge} \psi) + A \dot{\wedge} (A \dot{\wedge} \psi) = (dA + A \overset{\circ}{\wedge} A) \dot{\wedge} \psi \qquad (1.4)$$

is called *field strength.* Again, $\overset{\circ}{\wedge}$ denotes an exterior product together with a concatenation of the $\mathcal{G}$-values. The meaning of the field strength becomes clear if one identifies $v \rfloor w \rfloor F(p)$ (v and w are infinitesimal vectors) with $g(\gamma)$ for the closed path $\gamma = \partial(v \otimes w)$ around the infinitesimal 2-plane $v \otimes w$ attached at p, see figure 1.1

[*1.10*] All we introduced so far was motivated by the notion of an arbitrary connection that defines the comparison (1.1). This is now constraint by *the* fundamental principle of gauge theories: *All laws need to be invariant under local gauge transformations $g(x) \in G$, $x \in M$.* To ensure this we require all possible terms χ in laws to transform *covariant*, i.e. $\chi \to \chi' = g \cdot \chi$. When we consider a term $D\chi$ we also require its covariant transformation $D\chi \to (D\chi)' = g \cdot (D\chi)$ but at the same time $D\chi \to D(\chi') = D(g \cdot \chi)$, i.e. the derivative D and the local gauge $g(x)$ need to commute. This restricts the transformation behavior of the connection to

$$A' = \mathrm{Ad}_g(A) + g^{-1}(dg). \qquad (1.5)$$

[*1.11*] Having developed a local model of the observations, we need to find an appropriate mathematical formulation for a global model. We set up a principle bundle P with structure group G that represents all possible gauges at once and also enables a non-trivial *topology of a gauge* (see section 2.1.2). The linear connection and the field strength may now be defined as a $\mathcal{G}$-valued 1-form and 2-form, respectively, on this principle bundle P. The local field ψ is generalized as a section of the associated V-bundle which ensures the correct, covariant transformation behavior under local gauge transformations on P. If a local section σ of P is given, it is straightforward to locally identify the connection and the field strength on P with forms on M via the pull-back of σ. Note that *on the bundle* the connection transforms as $A \to A' = \mathrm{Ad}_g(A)$ under local gauge transformations $g(x) \in G$. (See appendix C.2.) We will refer to the bundle formalism only in the sections about symmetry breaking and topological charges. Elsewhere we keep to the local model.

[*1.12*] So far, we only introduced the basic fields we need but not any laws they yield. The common mechanism to produce such laws is to introduce the principle of least action, i.e. to introduce a lagrangian n-form and require the integral of this lagrangian to reach a minimum with respect to small variations of the fields the lagrangian depends on. You will find a short summary of the variational procedure in the appendix B. In gauge theories the lagrangian typically describes propagating gauge fields, i.e. it is proportional to a square term of F. For generality we only assume $\mathcal{L}_G = -\langle F \wedge H \rangle$, where we introduced the excitation $H = H(A, DA)$, which is a $\mathcal{G}$-valued 2-form, and the metric $\langle \ , \ \rangle$ in $\mathcal{G}$. In general, also the matter fields (i.e. the set Ψ of local fields $\psi_1, \psi_2, ...$) contribute a lagrangian $\mathcal{L}_{\rm mat} = \mathcal{L}_{\rm mat}(\Psi, D\Psi, A, DA)$ such that the complete, interacting theory is described by the total lagrangian $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{\rm mat}$. The action principle results in the field equation (following (B.8)):

$$0 = \frac{\delta \mathcal{L}}{\delta A} = D\frac{\partial \mathcal{L}_G}{\partial (DA)} + \frac{\partial \mathcal{L}_G}{\partial A} + \frac{\delta \mathcal{L}_{\rm mat}}{\delta A} = -DH + E + \Sigma \quad \in \Lambda^{n-1}(M, \mathcal{G}) \ , \tag{1.6}$$

where $E := \frac{\delta \mathcal{L}_G}{\delta A}$ and $\Sigma := \frac{\delta \mathcal{L}_{\rm mat}}{\delta A}$ are called *canonical energy-momentum current of the gauge fields and matter, respectively.*

[*1.13*] We require the lagrangian $\mathcal{L}$ to be invariant under the action of G. In the appendix, paragraph [*B.3*] and [*B.4*], we work out the variation for an internal transformation and a spacetime translation. The result are the Noether identities (B.12) and (B.20) for an internal current J and the canonical energy-momentum Σ.

Finally we mention the Bianchi identity $DF = 0$ which is the analog of the statement that a second application of the exterior derivative $d \circ d\cdot$ vanishes identically. (The Bianchi identity may also be understood as the field equation of the topological lagrangian $\langle F \wedge F \rangle$, which is the 1st Pontrjagin term.)

1.4 Symmetry breaking

This section is inspired by Trautman's review on gauge gravity [32], by O'Raifeartaigh's presentations of symmetry breaking in [5] and [29], by the interesting mechanism presented in [20], and by some discussions with F.W. Hehl. We first reformulate symmetry breaking emphasizing the group aspects and supplementing O'Raifeartaigh's contributions to this subject. This will provide us with a very intuitive and general formulation of symmetry breaking that is applicable to gravity. As a first step, we break the affine $A(n)$ symmetry down to the linear $GL(n)$ symmetry. Then, as a second step, we introduce an arbitrary metric which induces a splitting of the linear symmetry into antisymmetric and symmetric parts and, furthermore, may trigger a symmetry breaking down to the $O(1, n\text{–}1)$ symmetry.

Before we start, we briefly recall the presentation of symmetry breaking one can find in standard textbooks about quantum field theory (e.g. [8]) and point out some problems.

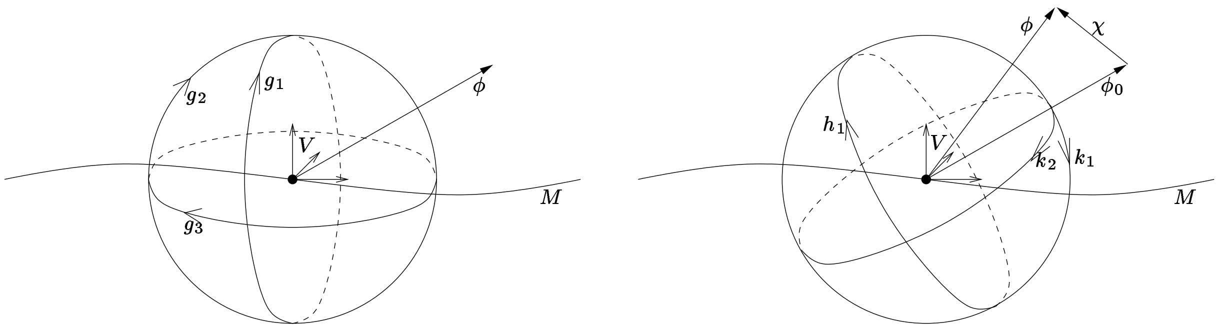


Figure 1.2: The left diagram represents the unbroken situation in a $G = O(3)$ gauge theory. The sphere indicates the three generators g_1, g_2, and g_3 of G that form the basis for the unbroken connection A. The free field ϕ has its values in V. In the right diagram we fixed the condensate ϕ_0 such that $\phi = \phi_0 + \chi$. The generators split into h_1 belonging to the stability group H and forming the basis for the residual connection B, and the generators k_1 and k_2 belonging to the quotient $K = G/H$ and forming the basis for the broken connection C.

First, one starts with an $O(3)$-invariant lagrangian, say:

$$\mathcal{L} = \frac{1}{2}\, D_\mu\phi_a D^\mu\phi^a + \frac{1}{2}\, m^2\, \phi_a\phi^a - \lambda(\phi_a\phi^a)^2 - \frac{1}{4} F_{\mu\nu a}{}^b F^{\mu\nu a}{}_b\ , \tag{1.7}$$
$$\text{where} \quad D_\mu\phi^i := \partial_\mu\phi^i + A_{\mu j}{}^i\phi^j\ , \quad F_{\mu\nu} := 2\,\partial_{[\mu}A_{\nu]} + A_\mu \circ A_\nu \in o(3)\ .$$

It includes a dynamical, a massive, and a quartic term of three scalar fields $\phi \in \mathbb{R}^3$, and a dynamical term of the gauge potential $A_\mu \in o(3)$. Second, one postulates a spontaneously selected vacuum state (or condensate) ϕ_0 (which is a minimum of the potential arising from the massive and quartic term) and decomposes $\phi = \phi_0 + \chi$. This splits the symmetry into a stability group $H = O(2)$ of ϕ_0 (also called little group or residual symmetry) and the quotient $K = G/H$, as illustrated in figure 1.2. Hence, the connection splits into $A_\mu = B_\mu + C_\mu$, with $A_\mu \in o(3)$, $B_\mu \in o(2)$, $C_\mu \in \mathcal{K} \sim \mathbb{R}^2$. Third, one chooses the local, unitary (or physical) gauge, i.e. $\phi \| \phi_0$ or $\chi \| \phi_0$. The consequent relation $(D_\mu\phi)_\perp = C_\mu(\phi_0 + \chi)$ means an absorbtion of orthogonal (Goldstone) modes by C, which now becomes massive if the square of ϕ_0 is non-trivial. Finally, one is left with one massive field χ, two massive vector fields C_μ, and one massless gauge potential B_μ.

Of course, this presentation is standard and not really incorrect. But it is not very strict and at first sight, perhaps, one might get misled and raise the following questions: What, if the symmetry is not large enough to align ϕ along some selected axis? If ϕ_0 is chosen and is not parallel to ϕ, how can an *orthogonal* gauge transformation make them parallel? And why should ϕ_0 be spatially constant? Having some insight one realizes that these questions are

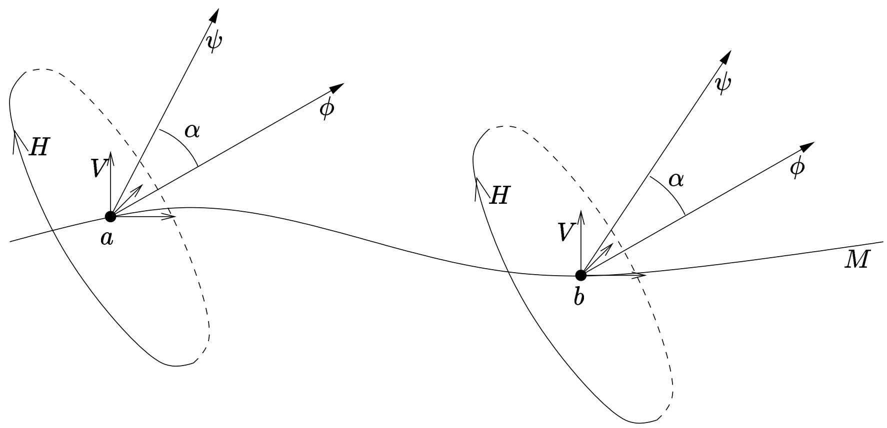


Figure 1.3: The Higgs field ϕ as a tool to compare states ψ at two points a and b in spacetime M.

not very deep and we will implicitely answer them below. O'Raifeartaigh's presentation of this subject [29] (based on an earlier presentation in [5]) is rather enlightening. He consideres a spatially dependent condensate $\phi_0 = \phi_0(x)$ and introduces a pure gauge $A^0_\mu = g^{-1}(\partial_\mu g)$ with $g = g(x) \in G$, $\phi_0(x) = g(x) \cdot \phi(0)$ such that $D^0_\mu \phi_0(x) = 0$ (as we will do similarly). However, in discussing the physical gauge in section 4, he considers the minimum of some function $f(a) = \langle \chi, a(\phi_0) \rangle$, where $a \in G$ represents a gauge transformation that rotates ϕ_0 such that its scalar product with $\chi = \phi - \phi_0$ becomes minimal, i.e. intuitively $\chi \| \phi_0$. Obviously, the transformation a acts only on ϕ_0 but not on χ. This is hardly understandable if one thinks that both, χ and ϕ_0, are elements of the same vector space on which G acts. As we will see below, ϕ_0 is rather an arbitrary representative of an orbit of G in this vector space.

1.4.1 The breaking mechanism

First, we develop a purely intuitive point of view before we formulate these insights mathematically.

[*1.14*] Following paragraph [*1.8*], the most basic idea of a connection is that it defines a comparison between neighboring fibres. Now suppose we have a spacetime with fibres V and a field $\phi(x) \in V$ and note that we also may use ϕ to compare states in different fibres: Let each fibre represent a $\mathbb{R}^3$, say, and let two states $\psi(a)$ and $\psi(b)$ at two (neighboring) points a and b in spacetime be given, see figure 1.3. We may now compare the states *with* ϕ *within* their fibres. As a result (neglecting absolute values) we get the angles $\alpha(a) = \angle(\psi(a), \phi(a))$ and $\alpha(b) = \angle(\psi(b), \phi(b))$ which are compared by simply considering

$\alpha(a) - \alpha(b)$. This is definitely an observational difference. Hence, the field ϕ *defines a comparison.* (No surprise that it entangles with the notion of a connection!) However, the comparison by ϕ is not as general as a connection. First, it is clear that this comparison is integrable, i.e. it gives no field strength. A local transformation $h \in H$ of the stability group H of ϕ applied on the fibre at a, say, does not change $\alpha(a) - \alpha(b)$. A local transformation $k \in K = G/H$, though, does change it. Hence, it is also clear that it entangles only with the $\mathcal{K}$-part of the connection, but not with the $\mathcal{H}$-part. If we seriously want to consider the comparison by ϕ in a gauge theory, we should *encode* it within the connection. This means to restrict the connection (i.e. fix the gauge) such that it always gives comparisons that are consistent with a comparison by ϕ. We give an exact formulation of these ideas:

[*1.15*] Let G be a gauge group with representation $\cdot$ on the vector space V, hence $G \cdot V \subseteq V$.

[*1.16*] Let W be a representative of the set of G-orbits V/G in V, i.e. we consider $W \subset V$ to be a specific but arbitrary representative of this quotient. We write $W \sim V/G$ in the following. Examples are $\big[G = SO(3),\ V = \mathbb{R}^3,\ W = \mathbb{R}\big]$, $\big[G = \{\mathrm{id}\},\ V,\ W = V\big]$, and $\big[G = U(1),\ V = \mathbb{C},\ W = \mathbb{R}^+\big]$.

[*1.17*] Let H be the *stabilizer* (or little group) of W defined by $H = \{g \in G :\ g \cdot W \subseteq W\}$. Note that $\forall(h \in H, w \in W) :\ h \cdot w = w$. $\big[$ Proof: For any h, the two elements $(h \cdot w)$ and w in V are G-equivalent. Since $W \sim V/G$ it follows that $(h \cdot w) = w \in W$ trivially. $\big]$ Let $\mathcal{H}$ be the Lie algebra of H and $\mathcal{K} = \mathcal{G}/\mathcal{H}$ such that $\mathcal{G} = \mathcal{H} \oplus \mathcal{K}$. The vector subspace $\mathcal{K}$ is, in general, not closed under the multiplication $[\ ,\ ]$ in $\mathcal{G}$.

[*1.18*] Let P be a principle bundle with structure group G, connection $A \in \Lambda^1(P, \mathcal{G})$, and a field $\phi : P \to V$ (a section in the associated vector bundle) defined on it. Now we restrict P to gauges (i.e. points in P) such that ϕ has its image in W, i.e. we restrict P to the *reduced principle bundle* $Q = \{p \in P : \phi(p) \in W\}$ with structure group H. The connection A naturally serves as a connection on Q if we consider only the $\mathcal{H}$-part. The $\mathcal{K}$-part of A, though, is not a connection on Q but a $\mathcal{K}$-valued 1-form on Q transforming homogeneously under the structure group H. We call ϕ *Higgs field.* To get more explicit results we switch to the local model in the following.

[*1.19*] Let M be a spacetime with a connection $A' \in \Lambda^1(M, \mathcal{G})$ and a Higgs field $\phi' : M \to V$ defined on it. We make a local gauge transformation $g(x)$ such that $\phi = g \cdot \phi'$ will have its values in W only:

$$\exists(g(x) \in G), \exists^{\text{unique}}(\phi(x) \in W) : \quad \phi(x) = g(x) \cdot \phi'(x)\ . \tag{1.8}$$

$\big[$ Proof: Since $W \sim V/G$, it is clear that there exists a $g \in G$ such that $g \cdot \phi' \in W$. ϕ is unique because W contains only *one* element in each orbit of G. $\big]$ We call this gauge *parallel gauge.* Restricting the theory to such a gauge is equivalent to reducing the principle bundle P to Q.

[*1.20*] We may write $g(x) = \exp(\delta(x) + \gamma(x))$ with $\delta \in \mathcal{H}$ and $\gamma \in \mathcal{K}$. We call $d\gamma(x)$ *Goldstone modes.* It is important to realize that, whenever we restrict the theory to a parallel gauge, then the degrees of freedom of $\phi' \in V$ split into degrees of freedom of $\phi \in W$ and $\gamma \in \mathcal{K}$. $\big[$ Proof: Since $\phi' = \exp(-\delta - \gamma) \cdot \phi$, the freedom of ϕ' splits into ϕ, δ, and γ. For given ϕ', though, δ is arbitrary since $\exp(\delta) \cdot \phi = \phi$, but ϕ and γ are unique. $\big]$ The connection transforms into

$$A = gA'g^{-1} + g(dg^{-1}) = gA'g^{-1} - d\delta - d\gamma \; . \tag{1.9}$$

[*1.21*] We may now split the $\mathcal{G}$-valued connection into a $\mathcal{H}$-part and a $\mathcal{K}$-part before and after the parallel gauge:

$$A' = A'_{\mathcal{H}} + A'_{\mathcal{K}} \; , \qquad \text{such that } gA'_{\mathcal{H}}g^{-1} \in \mathcal{H} \; , \; gA'_{\mathcal{K}}g^{-1} \in \mathcal{K} \; , \tag{1.10}$$

$$A = A_{\mathcal{H}} + A_{\mathcal{K}} \; , \qquad \text{such that } A_{\mathcal{H}} \in \mathcal{H} \; , \; A_{\mathcal{K}} \in \mathcal{K} \; . \tag{1.11}$$

Note that

$$A'_{\mathcal{H}} \cdot \phi' = g^{-1}(gA'_{\mathcal{H}}g^{-1}) \cdot (g\phi') = 0 \tag{1.12}$$

and, of course, $A_{\mathcal{H}} \cdot \phi = 0$. It follows

$$A = gA'g^{-1} - d\gamma - d\delta = gA'_{\mathcal{H}}g^{-1} + gA'_{\mathcal{K}}g^{-1} - d\gamma - d\delta \; , \tag{1.13}$$

$$A_{\mathcal{H}} = gA'_{\mathcal{H}}g^{-1} - d\delta \; , \tag{1.14}$$

$$A_{\mathcal{K}} = gA'_{\mathcal{K}}g^{-1} - d\gamma \; . \tag{1.15}$$

From now on, we only allow local gauge transformations $h(x) \in H$ that leave $\phi(x) \in W$ and finally reduce the principle bundle P to Q. Hence, $A_{\mathcal{H}}$ is our *residual connection* whereas the *broken connection* $A_{\mathcal{K}}$ transforms homogeneously under the *residual symmetry* H. We see that the Goldstone modes $d\gamma$ have been absorbed in $A_{\mathcal{K}}$. We find

$$g \cdot (D'\phi') \equiv D\phi = d\phi + A_{\mathcal{H}}\phi + A_{\mathcal{K}}\phi = d\phi + A_{\mathcal{K}}\phi \; , \tag{1.16}$$

where we used that $\mathcal{H} \cdot W = 0$. For the field strength $F = dA + A \overset{\circ}{\wedge} A$ we get

$$F = \big[dA_{\mathcal{H}} + A_{\mathcal{H}} \overset{\circ}{\wedge} A_{\mathcal{H}}\big] + \big[dA_{\mathcal{K}} + A_{\mathcal{K}} \overset{\circ}{\wedge} A_{\mathcal{K}}\big] + A_{\mathcal{H}} \overset{\circ}{\wedge} A_{\mathcal{K}} + A_{\mathcal{K}} \overset{\circ}{\wedge} A_{\mathcal{H}} \tag{1.17}$$

$$= F_{\mathcal{H}} + \big[dA_{\mathcal{K}} + A_{\mathcal{K}} \overset{\circ}{\wedge} A_{\mathcal{K}}\big] + A_{\mathcal{K}}{}^{a} \wedge A_{\mathcal{H}}{}^{b} \, [k_a, h_b] \; , \tag{1.18}$$

where $F_{\mathcal{H}} = [dA_{\mathcal{H}} + A_{\mathcal{H}} \overset{\circ}{\wedge} A_{\mathcal{H}}]$ is the *residual field strength* on Q and h_b and k_a are a basis in $\mathcal{H}$ and $\mathcal{K}$, respectively. We call $F - F_{\mathcal{H}}$ the *broken field strength.*

[*1.22*] If $\phi \in W$ has a non-vanishing, constant vacuum state $\phi_0 \in W$, $D_{\mathcal{H}}\phi_0 = (d+A_{\mathcal{H}})\phi_0 = d\phi_0 = 0$, $\phi = \phi_0+\chi$, $\chi \in W$, then we can assign a *vacuum mass matrix* $M_{ab} = \langle k_a \cdot \phi_0, k_b \cdot \phi_0\rangle$ to the broken connection $A_{\mathcal{K}}$:

$$(D\phi)^2 = (D\chi)^2 + M_{ab}({A_{\mathcal{K}}}^a \wedge {A_{\mathcal{K}}}^b) + 2\langle (A_{\mathcal{K}} \cdot \phi_0) \wedge D\chi\rangle \; . \tag{1.19}$$

This equation coincides with [29] eq (27).

1.4.2 Breaking the affine symmetry

Consider the principle bundle over a spacetime M with structure group $A(n)$ and algebra $\mathcal{A}(n)$. Suppose we are given a connection $\tilde{\tilde{\tilde{\Gamma}}}' \in \Lambda^1(M, \mathcal{A}(n))$ and a Higgs field $\tilde{\tilde{\tilde{\xi}}}' : M \to \mathbb{R}^n$, where $\mathbb{R}^n$ is affine. We find that $\mathbb{R}^n/A(n) \sim W = \{o\}$, where the affine point $o \in \mathbb{R}^n$ is an arbitrary representation of this equivalence class. The stabilizer H of W is the linear part $GL(n)$ of the affine group. We make the parallel gauge transformation $g = g(x)$, following (1.8), such that $\tilde{\tilde{\tilde{\xi}}} = g \cdot \tilde{\tilde{\tilde{\xi}}}' \stackrel{!}{=} o$, i.e. we translate $\tilde{\tilde{\tilde{\xi}}}'$ onto o. In the Moebius representation with respect to o we have

$$o = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; , \;\; \tilde{\tilde{\tilde{\xi}}}' = \begin{pmatrix} \xi \\ 1 \end{pmatrix} \; , \;\; g = \begin{pmatrix} \mathrm{id}^n & -\xi \\ 0 & 1 \end{pmatrix} \; , \;\; \tilde{\tilde{\tilde{\xi}}} = g \cdot \tilde{\tilde{\tilde{\xi}}}' = o \; , \tag{1.20}$$

$$\tilde{\tilde{\tilde{\Gamma}}}' =: \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} = \tilde{\tilde{\tilde{\Gamma}}}'_{\mathcal{G}l} + \tilde{\tilde{\tilde{\Gamma}}}'_{\mathcal{T}} \; ,$$

$$\text{where} \quad \tilde{\tilde{\tilde{\Gamma}}}'_{\mathcal{G}l} := \begin{pmatrix} X & -X \cdot \xi \\ 0 & 0 \end{pmatrix} \; , \quad \tilde{\tilde{\tilde{\Gamma}}}'_{\mathcal{T}} := \begin{pmatrix} 0 & X \cdot \xi + Y \\ 0 & 0 \end{pmatrix} \; , \tag{1.21}$$

$$\tilde{\tilde{\tilde{\Gamma}}} = g\tilde{\tilde{\tilde{\Gamma}}}' g^{-1} + g(dg^{-1}) = \begin{pmatrix} X & D_{\mathcal{G}l}\xi + Y \\ 0 & 0 \end{pmatrix} = \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{G}l} + \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{T}} \; ,$$

$$\text{where} \quad \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{G}l} := \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \; , \quad \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{T}} := \begin{pmatrix} 0 & D_{\mathcal{G}l}\xi + Y \\ 0 & 0 \end{pmatrix} \; . \tag{1.22}$$

Eq (1.21) refers to (1.10) and eq (1.22) refers to (1.11). Note that $g\tilde{\tilde{\tilde{\Gamma}}}'_{\mathcal{G}l} g^{-1} \in \mathcal{G}l(n)$ and $\tilde{\tilde{\tilde{\Gamma}}}'_{\mathcal{G}l} \cdot \tilde{\tilde{\tilde{\xi}}}' = 0$ as required by (1.12). We introduced the linear covariant derivative

$$D_{\mathcal{G}l}\xi = d\xi + X \cdot \xi \; . \tag{1.23}$$

Via (1.15) we identify the Goldstone modes

$$d\gamma = g\tilde{\tilde{\tilde{\Gamma}}}'_{\mathcal{T}} g^{-1} - \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{T}} = \begin{pmatrix} 0 & -d\xi \\ 0 & 0 \end{pmatrix} \; . \tag{1.24}$$

Eq (1.16) becomes

$$g(\tilde{\tilde{\tilde{D}}}' \tilde{\tilde{\tilde{\xi}}}') = \tilde{\tilde{\tilde{D}}} \tilde{\tilde{\tilde{\xi}}} = \tilde{\tilde{\tilde{D}}} o = do + \tilde{\tilde{\tilde{\Gamma}}} \cdot o = \begin{pmatrix} D_{\mathcal{G}l}\xi + Y \\ 0 \end{pmatrix} \; . \tag{1.25}$$

Following (1.17), the field strength reads

$$
\begin{aligned}
\tilde{\tilde{\tilde{F}}} &= d\tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{G}l} + \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{G}l} \overset{\circ}{\wedge} \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{G}l} + d\tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{T}} + \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{T}} \overset{\circ}{\wedge} \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{T}} + \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{G}l} \overset{\circ}{\wedge} \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{T}} + \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{T}} \overset{\circ}{\wedge} \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{G}l} \\
&= \left[d\tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{G}l} + \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{G}l} \overset{\circ}{\wedge} \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{G}l}\right] + \left[d\tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{T}} + 0 + \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{G}l} \overset{\circ}{\wedge} \tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{T}} + 0\right] \\
&= \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} , \quad \text{where}
\end{aligned}
$$

$$R := dX + X \overset{\circ}{\wedge} X \ , \tag{1.26}$$

$$T := D_{\mathcal{G}l}(D_{\mathcal{G}l}\xi + Y) \ . \tag{1.27}$$

The two zeros in the second line arise from the fact that the translational algebra acts non-trivial on affine points but trivial on vectors. So we found that the affine field strength splits into the residual field strength R and the broken field strength T that are called curvature and torsion, respectively.

1.4.3 Soldering the translational gauge

In this formulation it is quite easy to solder the translational gauge onto the manifold. It is straightforward to postulate the Higgs field $\tilde{\tilde{\tilde{\xi}}}' : M \to \mathbb{R}^n$ to be the local map of the manifold. Then, the Goldstone modes $-d\xi$ given in (1.24) (which, in our case, form the holonomic coframe!) will be absorbed by the translational gauge potential in the parallel gauge as eq (1.22) tells us:

$$\tilde{\tilde{\tilde{\Gamma}}}_{\mathcal{T}} = \begin{pmatrix} 0 & D_{\mathcal{G}l}\xi + Y \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} 0 & \vartheta \\ 0 & 0 \end{pmatrix} , \tag{1.28}$$

where the *holonomic* coframe $d\xi$ is absorbed and replaced by the *anholonomic* coframe

$$\vartheta = D_{\mathcal{G}l}\xi + Y \ , \tag{1.29}$$

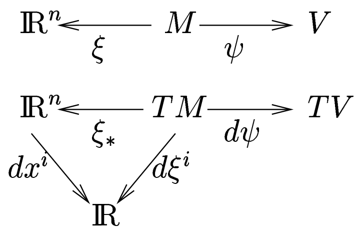
i.e. the affine-covariant derivative (1.25) of the local map. (Equation (1.29) is in analogy to [17] eq (3.2.11).) If we use ϑ as basis for all forms, we gain translational invariance for all forms. If we introduce a frame e_α in the vector space of $\mathbb{R}^n$, we may write in components:

$$\tilde{\tilde{\tilde{\xi}}}' = \begin{pmatrix} \xi^\alpha e_\alpha \\ 1 \end{pmatrix} , \quad g(\tilde{\tilde{\tilde{D}}}'\tilde{\tilde{\tilde{\xi}}}') = \tilde{\tilde{\tilde{D}}}\tilde{\tilde{\tilde{\xi}}} = \tilde{\tilde{\tilde{D}}}o = \begin{pmatrix} \vartheta^\alpha e_\alpha \\ 0 \end{pmatrix} \ . \tag{1.30}$$

The torsion (1.27) now reads

$$T = D_{\mathcal{G}l}\,\vartheta \quad \text{or} \quad T^\alpha = D_{\mathcal{G}l}\,\vartheta^\alpha \ . \tag{1.31}$$

What is the effect of this soldered translational gauge on physics? Any theory based on a (odd) lagrangian needs a spacetime metric $g = g_{\alpha\beta}\, d\xi^\alpha \otimes d\xi^\beta$. Replacing the exterior derivative of the local map $d\xi^\alpha$ by the affine-covariant ϑ^α while fixing the metric components thus means a change of the metric. This is why teleparallelism (where $g_{\alpha\beta} = o_{\alpha\beta} = \mathrm{diag}(+\ -\ -\ ...)$ is fixed) is so close to Einstein's theory. Also, per definition, the exterior derivative of any field $\psi :\ M \to V$ on the spacetime manifold is influenced (cf. [4] section 5.2):



$$d\psi := \left[d(\psi \circ \xi^{-1})\right] \circ \xi_* = \partial_\alpha(\psi \circ \xi^{-1})\ (dx^\alpha \circ \xi_*) = \partial_\alpha(\psi \circ \xi^{-1})\ d\xi^\alpha\ , \tag{1.32}$$
$$\text{or} \quad v\rfloor d\psi := (\xi_* v)\rfloor\left[d(\psi \circ \xi^{-1})\right] = \partial_\alpha(\psi \circ \xi^{-1})\ (\xi_* v)\rfloor dx^\alpha = \partial_\alpha(\psi \circ \xi^{-1})\ v\rfloor d\xi^\alpha$$

and thus

$$\tilde{\tilde{\tilde{D}}}\psi = \partial_\alpha(\psi \circ \xi^{-1})\, \tilde{\tilde{\tilde{D}}}\xi^\alpha = \partial_\alpha(\psi \circ \xi^{-1})\ \vartheta^\alpha\ . \tag{1.33}$$

Note here that the frame e_α and thus the coordinates x^α and the partial derivative ∂_α in $\mathbb{R}^n$ always refer to the so-called anholonomic frame. If the reader is familiar with the presentation in [17] or [15] he might wonder why we did not introduce different indices $_i$ and $_\alpha$ for a holonomic and anholonomic frame. The reason for Hehl et al. to do this is that they distinguish between a translational covariant version ϑ^α and a non-covariant version dx^i of the coframe. Both are related via an equation (cf. [15] section 2.33)

$$\vartheta^\alpha = \delta^\alpha_i dx^i + \Gamma_{\mathcal{T}}{}^\alpha \tag{1.34}$$

and a transformation defined by $\vartheta^\alpha = e_i{}^\alpha dx^i$. The symbol δ^α_i in equation (1.34) documents soldering by identifying these two types of indices but at the same time questions their distinction. Here, we write the affine-covariant and the non-covariant coframes as $\vartheta^\alpha = \tilde{\tilde{\tilde{D}}}\xi^\alpha$ and $d\xi^\alpha$ and thus need not to introduce different indices. Finally note that in this approach a dynamical term

$$(\tilde{\tilde{\tilde{D}}}\tilde{\tilde{\tilde{\xi}}})^2 = \vartheta^2 \sim \eta \tag{1.35}$$

of the local map (cf. (1.19)) in the lagrangian means a cosmological type term on the one hand and a *massive* term of the broken connection on the other.

In the following we replace the notations as follows:

$$X \to \Gamma$$
$$, D_{\mathcal{G}l} \to D\ , \tag{1.36}$$

and we may summarize that, in the language of paragraph [*1.18*], the initial principle bundle with affine structure group and connection $\tilde{\tilde{\Gamma}}'$ was reduced to a principle bundle with linear structure group and connection Γ. We gained the translational covariant coframe ϑ^α (transforming homogeneously under $GL(n)$) and will yield (1.33) whenever considering an exterior derivative. The field strength split into the torsion (1.31) and the curvature (1.26). A similar introduction of soldering and anholonomic coframes without a symmetry breaking mechanism is found in [15] sections 2.3-2.6.

1.4.4 *Splitting* the linear symmetry into antisymmetric and symmetric part

Before we consider a symmetry *breaking* of the linear group, let us generally investigate how the introduction of a metric induces the *splitting* of the $GL(n)$ symmetry into the antisymmetric part (the stabilizer of the metric) and the symmetric part. We are mainly interested in the splitting of the field strength. Consider the connection $\Gamma \in \Lambda^1(M, \mathcal{G}l(n))$ in matrix form $\Gamma: \ x^\alpha \to \Gamma_\beta{}^\alpha x^\beta$. With a given metric $h = h_{\alpha\beta}\, \vartheta^\alpha \otimes \vartheta^\beta$ we may raise and lower indices and define the antisymmetric and symmetric parts of a matrix as in table A.2:

$$\hat{\Gamma}_{\alpha\beta} := \Gamma_{[\alpha\beta]} \ , \quad \overset{\circ}{\Gamma}_{\alpha\beta} := \Gamma_{(\alpha\beta)} \ . \tag{1.37}$$

We define the nonmetricity by

$$Q_{\alpha\beta} := D h_{\alpha\beta} = d h_{\alpha\beta} + \Gamma^\gamma{}_\alpha h_{\gamma\beta} + \Gamma^\gamma{}_\beta h_{\alpha\gamma} = d h_{\alpha\beta} + 2\overset{\circ}{\Gamma}_{\alpha\beta} \ . \tag{1.38}$$

Following (1.17), we calculate

$$\begin{aligned} R_{\alpha\beta} &= d\hat{\Gamma}_{\alpha\beta} + \hat{\Gamma}_{\gamma\beta} \wedge \hat{\Gamma}_\alpha{}^\gamma + d\overset{\circ}{\Gamma}_{\alpha\beta} + \overset{\circ}{\Gamma}_{\gamma\beta} \wedge \overset{\circ}{\Gamma}_\alpha{}^\gamma + \hat{\Gamma}_{\gamma\beta} \wedge \overset{\circ}{\Gamma}_\alpha{}^\gamma + \overset{\circ}{\Gamma}_{\gamma\beta} \wedge \hat{\Gamma}_\alpha{}^\gamma \\ &= d\hat{\Gamma}_{[\alpha\beta]} + \hat{\Gamma}_{\gamma[\beta} \wedge \hat{\Gamma}_{\alpha]}{}^\gamma + d\overset{\circ}{\Gamma}_{(\alpha\beta)} + \overset{\circ}{\Gamma}_{\gamma[\beta} \wedge \overset{\circ}{\Gamma}_{\alpha]}{}^\gamma + \hat{\Gamma}_{\gamma(\beta} \wedge \overset{\circ}{\Gamma}_{\alpha)}{}^\gamma + \overset{\circ}{\Gamma}_{\gamma(\beta} \wedge \hat{\Gamma}_{\alpha)}{}^\gamma \ . \end{aligned} \tag{1.39}$$

We found the symmetries in the second line by the simple calculations in the appendix, paragraph [*A.7*]. This means that the field strength splits into the antisymmetric and symmetric part

$$\hat{R}_{\alpha\beta} := R_{[\alpha\beta]} = d\hat{\Gamma}_{\alpha\beta} + \hat{\Gamma}_{\gamma\beta} \wedge \hat{\Gamma}_\alpha{}^\gamma + \overset{\circ}{\Gamma}_{\gamma\beta} \wedge \overset{\circ}{\Gamma}_\alpha{}^\gamma \ , \tag{1.40}$$

$$\begin{aligned} \overset{\circ}{R}_{\alpha\beta} &:= R_{(\alpha\beta)} = d\overset{\circ}{\Gamma}_{\alpha\beta} + \hat{\Gamma}_{\gamma\beta} \wedge \overset{\circ}{\Gamma}_\alpha{}^\gamma + \overset{\circ}{\Gamma}_{\gamma\beta} \wedge \hat{\Gamma}_\alpha{}^\gamma \\ &\overset{(A.33)}{=} d\overset{\circ}{\Gamma}_{\alpha\beta} + \hat{\Gamma}^\gamma{}_\beta \wedge \overset{\circ}{\Gamma}_{\alpha\gamma} + \hat{\Gamma}^\gamma{}_\alpha \wedge \overset{\circ}{\Gamma}_{\gamma\beta} = \hat{D}\overset{\circ}{\Gamma}_{\alpha\beta} = \frac{1}{2}(\hat{D}Q_{\alpha\beta} - \hat{\Gamma} \cdot dh_{\alpha\beta}) \ . \end{aligned} \tag{1.41}$$

We will keep in mind the remarkable last relation.

1.4.5 *Breaking* the linear symmetry

The following ideas are very close to those in [32] and [20]. Consider the principle bundle with linear structure group $GL(n)$, algebra $\mathcal{G}l(n)$, and connection Γ'. Suppose we are given a Higgs field $h' : M \to V := T^*M \otimes T^*M$, i.e. a metric. This field transforms under $g \in GL(n)$ as

$$h(v,w) = (g \cdot h')(v,w) = h'(g \cdot v, g \cdot w) \quad \text{or} \quad h = g \cdot h' = h' \circ (g \otimes g) \ . \tag{1.42}$$

The transformation under the algebra $X \in \mathcal{G}l(n)$ is derived from (C.3):

$$X \cdot h' = h' \circ (X \otimes \mathrm{id}) + h' \circ (\mathrm{id} \otimes X) \quad \text{or} \tag{1.43}$$

$$X \cdot h'_{\alpha\beta} = X^\gamma{}_\alpha \, h_{\gamma\beta} + X^\gamma{}_\beta \, h_{\alpha\gamma} = 2\overset{\circ}{X}_{\alpha\beta} \ , \tag{1.44}$$

what we already used in (1.38). We find that the space $W \sim V/GL(n)$ only includes one representative for each signature. This is because the set of all metrics with a fixed signature is an orbit of $GL(n)$ and hence belongs to the same equivalence class in $V/GL(n)$. Thus, for a given signature, we choose $o \in W$ to be the representative of all metrics with this signature. To simplify things, we restrict ourselves to metrics with Lorentzian signature $(+ \ - \ - \ ...)$. Then $W = \{o\}$, where o is an arbitrary metric with Lorentzian signature. The stabilizer H of W is the Lorentz group $O(1, n{-}1)$. The algebra $\mathcal{G}l(n)$ thus splits into an antisymmetric part $\mathcal{H} = o(1, n{-}1)$ and a symmetric part $\mathcal{K}$ as we prepared in the previous section. It is easy to verify $\Gamma'_\mathcal{H} = \hat{\Gamma}'$ and $\Gamma'_\mathcal{K} = \overset{\circ}{\Gamma}'$ by comparing the relation $\Gamma' \cdot h'_{\alpha\beta} = 2\overset{\circ}{\Gamma}'_{\alpha\beta}$ with (1.12). We perform the parallel gauge transformation $g = g(x)$, following (1.8), such that $h = g \cdot h' \overset{!}{=} o$. If, in components, $g : \ x^\alpha \to g_\beta{}^\alpha \, x^\beta$ and $g^{-1} : \ x^\alpha \to g^\alpha{}_\beta \, x^\beta$, then the fields transform as

$$h_{\alpha\beta} = g^\gamma{}_\alpha \, g^\delta{}_\beta \, h'_{\gamma\delta} = o_{\alpha\beta} \ , \tag{1.45}$$

$$\Gamma_\alpha{}^\beta = \left(g^{-1}\Gamma' g + g(dg^{-1})\right)_\alpha{}^\beta = g^\beta{}_\delta \, \Gamma'_\gamma{}^\delta \, g_\alpha{}^\gamma + g_\gamma{}^\beta \, dg^\gamma{}_\alpha \ . \tag{1.46}$$

In this gauge, the nonmetricity (1.38) reads (cf. (1.16))

$$Q_{\alpha\beta} = Dh_{\alpha\beta} = Do_{\alpha\beta} = \Gamma \cdot o_{\alpha\beta} = 2\overset{\circ}{\Gamma}_{\alpha\beta} \ , \tag{1.47}$$

where $\overset{\circ}{\Gamma}$ is symmetrized with respect to the fixed metric $o_{\alpha\beta}$. By using (1.40) and (1.41), together with $dh = do = 0$ in the parallel gauge, the field strength R may be expressed as

$$R_{\alpha\beta} = \hat{R}_{\alpha\beta} + \overset{\circ}{R}_{\alpha\beta} \ , \quad \overset{\circ}{R}_{\alpha\beta} = \frac{1}{2}\hat{D}Q_{\alpha\beta} \ , \quad \hat{R}_{\alpha\beta} = R_{\mathcal{H}\alpha\beta} + \overset{\circ}{\Gamma}_{\gamma\beta} \wedge \overset{\circ}{\Gamma}_\alpha{}^\gamma \ , \tag{1.48}$$

$$R_{\mathcal{H}\alpha\beta} := d\hat{\Gamma}_{\alpha\beta} + \hat{\Gamma}_{\gamma\beta} \wedge \hat{\Gamma}_\alpha{}^\gamma \ . \tag{1.49}$$

We are left with an $O(1,n-1)$ gauge theory with connection $\hat{\Gamma}_{\alpha\beta}$, a fixed Lorentzian metric $o_{\alpha\beta}$, and an independent field $Q_{\alpha\beta} = 2\overset{\circ}{\Gamma}_{\alpha\beta}$ (transforming homogeneously under $O(1,n-1)$) which is the broken connection. In fact, if a lagrangian included a dynamical term $(Dh_{\alpha\beta})^2 = (Q_{\alpha\beta})^2$ (cf. (1.19)) and the field strength square $(\overset{\circ}{R}_{\alpha\beta})^2 = \frac{1}{4}(\hat{D}Q_{\alpha\beta})^2$, then $\overset{\circ}{\Gamma}_{\alpha\beta} = \frac{1}{2}Q_{\alpha\beta}$ has become massive! In the rest of this work, this kind of splitting and breaking the linear symmetry will not be explicitly relevant, but it helps to motivate chapter 3. This discussion might also help to judge a short discussion of such a parallel gauge in [15] section 3.5.1.

See table 1.1 for a summary of this section.

1.5 Gravity as a gauge theory

After this long and general considerations, we will finally introduce a gauge theory of gravity. We already introduced the general formalism of this theory in the previous sections. Accordingly, we can now concentrate on the motivation and on the dynamics of the theory. In later sections, we will also consider special cases of the general theory, including Einstein gravity.

Any motivation of a theory will necessary touch the problems we discussed in section 1.2. However, since this is not supposed to be an epistemological work, we do not try to understand how physicists (and mathematicians!) could develop a language and notions like spacetime, mass, energy-momentum, spin, etc. We only recognize that these notions correspond to the Casimir operators and the current of a group – the Poincaré group – and that observers seem to be distinguished by their location and orientation in spacetime. The latter directly leads to the spacetime diffeomorphisms as a symmetry group. In a local approximation of spacetime, i.e. the affine tangent space A_pM, this group is the affine group, having the Poincaré group as subgroup. Hence, we ought to be successful in formulating a gauge theory of the affine group. We introduced such a theory in sections 1.4.2 and 1.4.3 and enriched the formalism by splitting and breaking the linear part of the symmetry in section 1.4.4 and 1.4.5. Table 1.2 summarizes the fields involved in this formalism.

The dynamics of this theory is specified by a lagrangian $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{\text{mat}}$, following paragraph [*1.12*], such that $\mathcal{L}_G$ has the excitations

$$-\frac{\partial \mathcal{L}_G}{\partial T^\alpha} = H_\alpha \ , \quad -\frac{\partial \mathcal{L}_G}{\partial R_\alpha{}^\beta} = H^\alpha{}_\beta \ , \quad -\frac{\partial \mathcal{L}_G}{\partial Q_{\alpha\beta}} = \frac{1}{2} M^{\alpha\beta} \ . \tag{1.50}$$

	section 1.4.1	section 1.4.3	section 1.4.5
original symmetry	G	affine group $A(n)$	linear group $GL(n)$
residual symmetry	H	linear group $GL(n)$	orthogonal group $O(1, n-1)$
residual connection	$A_{\mathcal{H}}$	**linear connection** Γ	antisym. connection $\hat{\Gamma}$
residual covariant derivative	$D_{\mathcal{H}} = d + A_{\mathcal{H}}$	$D_{\mathcal{G}l} = d + \Gamma$	$\hat{D} = d + \hat{\Gamma}$
broken (massive?) connection	$A_{\mathcal{K}}$	**anholonomic coframe** ϑ (1.28)	sym. connection, **nonmetricity** $\overset{\circ}{\Gamma} = \frac{1}{2}Q$
residual field strength	$F_{\mathcal{H}} = dA_{\mathcal{H}} + A_{\mathcal{H}} \overset{\circ}{\wedge} A_{\mathcal{H}}$	**curvature** R (1.26)	$R_{\mathcal{H}} = d\hat{\Gamma} + \hat{\Gamma} \overset{\circ}{\wedge} \hat{\Gamma}$ (1.49)
broken field strength	$F - F_{\mathcal{H}}$ (1.17)	**torsion** T (1.31)	$\overset{\circ}{\Gamma} \overset{\circ}{\wedge} \hat{\Gamma} + \frac{1}{2}\hat{D}Q$ (1.48)
Goldstone modes	$d\gamma$	**holonomic coframe** $-d\xi$ (1.24)	?
Higgs field	ϕ'	local affine map $\tilde{\tilde{\tilde{\xi}}}'$	Lorentzian metric h'
covariant derivative of the Higgs field	$D\phi = d\phi + A_{\mathcal{K}}\phi$ (1.16)	**anholonomic coframe** $\tilde{\tilde{\tilde{D}}}\tilde{\tilde{\tilde{\xi}}} = \vartheta$ (1.25), (1.29)	**nonmetricity** $Dh = Q = 2\overset{\circ}{\Gamma}$ (1.47)
origin of massive terms (for $A_{\mathcal{K}}$) in the lagrangian	dynamical term $(D\phi)^2$ (1.19)	**cosmological type term** $(\tilde{\tilde{\tilde{D}}}\tilde{\tilde{\tilde{\xi}}})^2 = \eta$ (1.35)	$(Dh)^2 = Q^2$

Table 1.1: We summarize our discussion of symmetry breaking in affine gauge theories of gravity. Note that section 1.4.5 assumes the parallel gauge g such that the metric is constant, $h = g \cdot h' = o$, and thus $dh = 0$. Without this gauge, the linear symmetry is not really *broken* and only the general relations in section 1.4.4 hold. Further explanations can be found in the text.

	potential	strength	momentum	gauge current	matter current
symbolically	A	F	$H := -\frac{\partial \mathcal{L}_G}{\partial F}$	$E := \frac{\partial \mathcal{L}_G}{\partial A}$	$\Sigma := \frac{\delta \mathcal{L}_{\rm mat}}{\delta A}$
translations	ϑ^α	T^α	H_α	E_α	Σ_α
linear transf.	$\Gamma_\alpha{}^\beta$	$R_\alpha{}^\beta$	$H^\alpha{}_\beta$	$E^\alpha{}_\beta$	$\Delta^\alpha{}_\beta$
metric	$g_{\alpha\beta}$	$Q_{\alpha\beta}$	$\frac{1}{2}M^{\alpha\beta}$	$\frac{1}{2}m^{\alpha\beta}$	$\frac{1}{2}\sigma^{\alpha\beta}$

Table 1.2: All the relevant quantities of metric-affine gauge theory of gravity are collected. The metric g, displayed in the last line, is not really a gauge potential but the associated quantities are defined analogously.

By the methods described in appendix B we find the field equations

$$\frac{\delta \mathcal{L}_{\rm mat}}{\delta \psi} = 0 \,, \tag{1.51}$$

$$DM^{\alpha\beta} - m^{\alpha\beta} = \sigma^{\alpha\beta} \,, \qquad \text{ZEROTH} \,, \tag{1.52}$$

$$DH_\alpha - E_\alpha = \Sigma_\alpha \,, \qquad \text{FIRST} \,, \tag{1.53}$$

$$DH^\alpha{}_\beta - E^\alpha{}_\beta = \Delta^\alpha{}_\beta \,, \qquad \text{SECOND} \,, \tag{1.54}$$

cf. [17] section 5.5. The invariance of the lagrangian under spacetime translations and linear transformations yields two Noether identities

$$D\Sigma_\alpha \stackrel{*}{=} (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \Delta^\beta{}_\gamma - \frac{1}{2}(e_\alpha \rfloor Q_{\beta\gamma}) \wedge \sigma^{\beta\gamma} \,, \tag{1.55}$$

$$D\Delta^\alpha{}_\beta \stackrel{*}{=} g_{\beta\gamma}\sigma^{\alpha\gamma} - \vartheta^\alpha \wedge \Sigma_\beta \,, \tag{1.56}$$

which are presented here in the *weak* form, i.e. for $\frac{\delta \mathcal{L}_{\rm mat}}{\delta \psi} \stackrel{*}{=} 0$. A general derivation is given in [17] section 5.2.

1.5.1 Special cases of MAG

The theory we developed so far may be restricted to theories of smaller symmetries by constraining some fields or by introducing symmetry breaking. A natural constraint on the gauge potential is induced by restricting the linear part of the gauge group. (The translations may not be restricted because this would destroy the coupling to the canonical energy-momentum.) The connection $\Gamma_{\alpha\beta}$ of the linear group may be split (via a given metric) into its trace, its traceless-symmetric, and its antisymmetric part:

$$\Gamma_{\alpha\beta} = \frac{1}{n}\, g_{\alpha\beta}\, \Gamma_\gamma{}^\gamma + \not\!\!\Gamma_{\alpha\beta} + \Gamma_{[\alpha\beta]}\,, \quad \not\!\!\Gamma_{\alpha\beta} := \Gamma_{(\alpha\beta)} - \frac{1}{n}\, g_{\alpha\beta}\, \Gamma_\gamma{}^\gamma\,. \tag{1.57}$$

Dropping the *dilation* $\Gamma_\gamma{}^\gamma$ means considering only a volume preserving symmetry; dropping the *shear* $\not\!\!\Gamma_{\alpha\beta}$ leads to the so-called Weyl-invariance; dropping both leads to the Poincaré gauge theory; and dropping all $\Gamma_{\alpha\beta}$'s leads to teleparallel theories. Furthermore, the metric or nonmetricity may be constrained. In section 1.4.5 we also found a way to arrive at a Poincaré theory with fixed metric and additional fields.

The teleparallel case with fixed metric components $o_{\alpha\beta}$ (or, equivalently, vanishing nonmetricity) is of special interest because, for a special lagrangian, the theory is equivalent to Einstein gravity. In this case, only the anholonomic coframe ϑ^α is a free variable (besides the matter fields) and FIRST (1.53) represents the field equation. The *geometry* of the underlying spacetime may be described as Riemannian with respect to the metric $g = o_{\alpha\beta}\, \vartheta^\alpha \otimes \vartheta^\beta$, its Levi-Civita connection $K_\alpha{}^\beta$ (called contortion from the teleparallel point of view), and its Riemannian curvature $R_\alpha{}^\beta$. The Levi-Civita connection of the Riemannian geometry is related to torsion in the teleparallel formalism via $T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta$. For the special choice

$$H_\alpha = -\frac{1}{2} K^{\mu\nu} \wedge \eta_{\alpha\mu\nu} \tag{1.58}$$

of the excitation, the field equation FIRST (1.53) is equivalent to the Einstein equation of the described Riemannian geometry, see e.g. [15]. Also the Noether identity (1.55) (see also (B.21)) of the teleparallel theory, i.e. $0 = d\,\Sigma_\alpha - (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta$, is equivalent to the Noether identity $0 = d\,\Sigma_\alpha + K^\beta{}_\alpha \Sigma_\beta$ of the Einstein theory provided the energy-momentum tensor is symmetric, i.e. $\vartheta_{[\gamma} \wedge \Sigma_{\beta]} = 0$:

$$\begin{aligned} \left[e_\alpha \rfloor T^\beta\right] \wedge \Sigma_\beta &= \left[e_\alpha \rfloor (K^\beta{}_\gamma \wedge \vartheta^\gamma)\right] \wedge \Sigma_\beta = (e_\alpha \rfloor K^\beta{}_\gamma)\, \vartheta^\gamma \wedge \Sigma_\beta - K^\beta{}_\alpha \wedge \Sigma_\beta \\ &= (e_\alpha \rfloor K^{\beta\gamma})\, \vartheta_{[\gamma} \wedge \Sigma_{\beta]} - K^\beta{}_\alpha \wedge \Sigma_\beta = -K^\beta{}_\alpha \wedge \Sigma_\beta\,. \end{aligned} \tag{1.59}$$

1.5.2 The Kaluza-Klein theory

The Kaluza-Klein theory unifies Einstein gravity and electromagnetism. In the original approach, the theory is formulated as Einstein gravity in 5 dimensions. Given a 5D metric $\tilde{g}$, we define the electromagnetic potential A and the 4D metric g by formally breaking isotropy and selecting the 5th dimension:

$$\tilde{g}_{ab} = \begin{pmatrix} \tilde{g}_{ij} & \tilde{g}_{5i} \\ \tilde{g}_{j5} & \tilde{g}_{55} \end{pmatrix} = \begin{pmatrix} g_{ij} + A_i A_j & A_i \\ A_j & \tilde{g}_{55} \end{pmatrix} , \quad a,b = 0,1,2,3,5 , \quad i,j = 0,1,2,3 . \tag{1.60}$$

Investigating the 5-dimensional Einstein-Hilbert lagrangian, we find

$$\sqrt{-\det \tilde{g}_{ab}}\ \tilde{R} = \sqrt{-\det g_{ij}} \left(R + \frac{1}{4}\, F_{ij} F^{ij} \right) , \tag{1.61}$$

where $\tilde{R}$ is the 5D curvature scalar of $\tilde{g}_{ab}$, R the 4D curvature scalar of g_{ij}, and $F_{ij} = 2\,\partial_{[i} A_{j]}$ the electromagnetic field strength. The variation with repect to $\tilde{g}_{ij}$ leads to the ordinary 4-dimensional Einstein field equation with the electromagnetic energy-momentum as source. The variation with respect to $\tilde{g}_{5i} = A_i$ leads to the inhomogeneous Maxwell field equation on a curved background. The variation with respect to $\tilde{g}_{55}$, though, leads to a field equation that goes beyond the unified Einstein-Maxwell theory and may in fact be physically wrong. In conventional formulations of Kaluza-Klein theories the problem is bypassed by postulating that $\tilde{g}_{55} = 1$ identically and may not be varied.

The gauge theoretical analogue to this formalism is the 5-dimensional teleparallelism with excitation (1.58). It is very nice to recognize that the 5th covector $\vartheta^{\hat{5}}$ (which represents the gauge potential of the 5th translation) plays the role of the electromagnetic potential A. Hence, $T^{\hat{5}}$ is the electromagnetic field strength. In 5 dimensions, the special choice of the teleparallel lagrangian (1.58) has the explicit form

$$H_{5D} = {}^{\star(1)}T - 3\,{}^{\star(2)}T + \frac{5}{2}\,{}^{\star(3)}T . \tag{1.62}$$

Here, ${}^{(1)}T$, ${}^{(2)}T$, and ${}^{(3)}T$ denote the irreducible parts of the torsion with respect to a linear symmetry with metric as given in [17] appendix B.

1.6 Dimensional analysis of gauge theories

We found the essential fields involved in a gauge theory of a Lie group G (with algebra $\mathcal{G}$) to be the connection A, the field strength F, the excitation H, the lagrangian $\mathcal{L}$, and the Noether current Σ. The connection is introduced as a $\mathcal{G}$-valued 1-form on the principle bundle or, locally, as a $\mathcal{G}$-valued 1-form on spacetime, i.e. $A \in \Lambda^1(M, \mathcal{G})$. It yields the covariant derivative $D = d + A$.

By its very definition, the exterior differentiation operator d is dimensionless, $[d] = 1$. Hence we also require the connection to be dimensionless, $[A] = 1$. Now we need to give *exactly two* definitions in order to find all the remaining dimensions. First, we *choose to define* the dimension of a lagrangian $\mathcal{L}$ to be $[\mathcal{L}] = \hbar$. (In the context of a classical gauge theory $\hbar$, is merely the name of a dimension as introduced here. However, thinking of Huygen's principle and the path integral method, one may also call $\hbar$ a *phase*/2π *unit.*) And second, we define the basis elements λ_a of the algebra $\mathcal{G}$ to have the dimension $[\lambda_a] = g/\hbar$. (Again, so far g is merely the name of a dimension introduced here. However, in the case of electromagnetism, it may be replaced by the *unit* e.) Now it is easy to display the dimensions of the components of $A \equiv A^a\,\lambda_a \equiv A_i{}^a\,\lambda_a\,dx^i$ and $F \equiv F^a\,\lambda_a \equiv \frac{1}{2}F_{ij}{}^a\,\lambda_a\,dx^i \wedge dx^j$. You will find them in table 1.3. We then read off the dimension of the excitation $H \equiv H^a\,\lambda_a \equiv \frac{1}{2}H_{ij}{}^a\,\lambda_a\,dx^i \wedge dx^j$ and of the Noether current $\Sigma_a := \delta\mathcal{L}/\delta A^a$. For consistency, the dimension of the metric $\langle\ ,\ \rangle$ in $\mathcal{G}$ has to be $[\langle\ ,\ \rangle] = \hbar^2/g^2$. It follows $[\langle\lambda_a, \lambda_b\rangle] = 1$. The dimension of $[H]/[F] = g^2/\hbar$ may be interpreted as the dimension of the coupling constant $1/\kappa$ of a dynamical lagrangian with $H \approx (1/\kappa)\ {}^\star F$.

In the case of electrodynamics, we only have one index $a = 0$ and we set $[\lambda_0] = e/\hbar$. We see that the *algebra components* F^a, H^a, and Σ_a carry the conventional dimensions, whereas the dimensions of the fields F, H, Σ are more unfamiliar. In the case of a translational gauge theory, we assign the dimension $1/\ell$ to the generators (where ℓ means *length*) and find that $[1/\kappa] = \hbar/\ell^2$. Since this dimensionality includes a length dimension perturbation theory does not work. When embedding electrodynamics in an extra dimension à la Kaluza-Klein, the $U(1)$ gauge is directly represented by the translation along the 5th dimension. We can introduce a length *unit* ℓ_5 of this 5th dimension by identifying $e/\hbar = 1/\ell_5$. This is a geometrical interpretation of the electric unit e as *phase*/2π *per length.* Besides, if the 5th dimension is $U(1)$ with perimeter L_5, it seems natural that this 'phase/2π per length'-unit e is quantized in quanta of $\hbar/L_5$. Thus, we may assume that the perimeter of $U(1)$ is $L_5 = \ell_5 = \hbar/e$.

We want to point out again that any dimensional system of a gauge theory (as long as all generators have the same dimension) may be spanned by exactly two definitions, e.g. those for $[\mathcal{L}]$ and $[\lambda_a]$. This is the reason why every column in table 1.3 includes exactly two dimensions (or units).

1.7 Summary and a speculative idea

As a result of this chapter we introduced gravity as a gauge theory and finally formulated the *metric-affine gauge theory of gravity* (MAG) and special cases of this theory in section 1.5. Table 1.4 displays the important analogy between *internal* and *external* gauge theories, i.e. between Yang-Mills type theories and gauge theories of gravity. Table 1.1 presents

	in general	in electrodynamics	in translational gauge theories
$[\mathcal{L} := F^a \wedge H_a] \quad =:$	$\hbar$	Wb C	$\hbar$
$[\lambda_a] \quad =:$	$g/\hbar$	1/Wb	$1/\ell$
$[A = A^a \lambda_a] \quad \equiv$	1	1	1
$[F = F^a \lambda_a]$	1	1	1
$[H = H^a \lambda_a]$	$g^2/\hbar$	C/Wb	$\hbar/\ell^2$
$[A^a = A_i{}^a\, dx^i]$	$\hbar/g$	Wb	ℓ
$[F^a = \frac{1}{2}\, F_{ij}{}^a\, dx^i \wedge dx^j]$	$\hbar/g$	Wb	ℓ
$[H^a = \frac{1}{2}\, H_{ij}{}^a\, dx^i \wedge dx^j]$	g	C	$\hbar/\ell$
$[\Sigma_a = \delta\mathcal{L}/\delta A^a]$	g	C	$\hbar/\ell$
$[\langle\ ,\ \rangle] = 1/[\lambda]^2$	$\hbar^2/g^2$	Wb^2	ℓ^2
$[\langle F, H\rangle]$	$\hbar$	Wb C	$\hbar$
$[1/\kappa] = [H]/[F] = [H^a]/[F^a]$	$g^2/\hbar$	C/Wb	$\hbar/\ell^2$
$[\mathcal{E}] = [\mathcal{M}] = [F]$	1	1	1
$[\mathcal{E}^a] = [\mathcal{M}^a] = [F^a]$	$\hbar/g$	Wb	ℓ
$[\mathcal{I}]$	g	C	$\hbar/\ell$

Table 1.3: The table displays the dimensions of essential fields and objects involved in a gauge theory. In particular, it gives the SI-units in the case of electrodynamics and the dimensions for a translational gauge theory. We stress that the first two rows in this table are definitions, the third is an identity, and the rest is a consequence. The last block includes the dimensions of monopole and topological charges. The SI-units used in electrodynamics are C=Coulomb and Wb=Weber. We have the lagrangian $\mathcal{L}$, group generators λ_a, gauge potential A, field strength F, excitation H, Noether current Σ, algebra metric $\langle\ ,\ \rangle$, coupling constant $1/\kappa$, quasi-electric and -magnetic charge $\mathcal{E}$ and $\mathcal{M}$, and elementary charge $\mathcal{I}$.

theory	gauge group	connection	field strength
general gauge theory	semi-simple Lie group G	$A \in \Lambda^1(M, \mathcal{G})$	$F = D\Gamma \in \Lambda^2(M, \mathcal{G})$
electrodynamics	$U(1)$	A	$F = dA$
(*non-physical*)	affine group	$\stackrel{\approx}{\Gamma}$	$\stackrel{\approx}{R}$
affine gauge theory	soldered affine group	$\Gamma_\alpha{}^\beta$, ϑ^α	$R_\alpha{}^\beta$, T^α
teleparallelism	soldered translations	ϑ^α	$T^\alpha = d\vartheta^a$

Table 1.4: Overview of gauge formalisms: Gravity may be described by formulating a gauge theory of the affine group. However, one has to ensure that the group, i.e. the corresponding Lie-algebra valued connection, acts on spacetime. This is done by splitting the connection into a linear part $\Gamma_\alpha{}^\beta$ (with matrix indices ${}_\alpha{}^\beta$ that act on the basis e_α of the local tangent space) and an inhomogeneous part ϑ^α (that replaces the holonomic coframe $d\xi^\alpha$ and thereby realizes a translational gauge). The field strength splits into the curvature $R_\alpha{}^\beta$ and the torsion T^α. Discarding the linear gauge ($\Gamma_\alpha{}^\beta \equiv 0$), the theory reduces to teleparallelism.

our result of the discussion of symmetry breaking in gauge theories of gravity and our dimensional analysis of gauge theories is summarized by table 1.3. Some ideas the author wanted to stress are (1) the philosophical considerations concerning the constitution of objects, (2) the *motivation* of the connection as a comparison operator rather than some object that rescues local gauge invariance, (3) the meaning of symmetry breaking and its exact formulation, and (4) the motivation of soldering as an effect of the breaking of translational invariance by the spacetime's local map.

We want to add a speculation at this place. It concerns the soldering process and the role of the local map ξ of spacetime as a Higgs field. First, one should note that it is a rather new point of view to consider the differentiation to be defined via a *dynamical* Higgs field. The explicit form of the symmetry breaking condensate may decide on the nature of some dimensions. If, e.g., the Higgs field ξ happens to be constant along some dimensions, then these dimensions have no extensions because the coframe $d\xi$ along these dimensions is identically zero. One might say that such dimensions are not unfolded whereas others (perhaps four in number!) are unfolded. This means some kind of fibring of the initial manifold. So, the point of this speculation might be a *dynamical* mechanism that explains the unfolding of four dimensions and the soldering of the affine gauge on these dimensions, i.e. spacetime and gravity.

Chapter 2

The charge concept in gauge theories

2.1 Introduction

What is a charge? In general is seems plausible to define a charge to be a specific and invariant property of a particle (usually given by one number, perhaps an integer). Since in gauge theories we take particles to be elements in a representation of the symmetry, we are directly led to the most basic notion of a charge, the *elementary charge*, classifying the representation of the symmetry the particle is an element of. But also a specific property of the gauge field which a particle *necessarily* induces may be considered as a charge. Such is, e.g., the monopole character of the electromagnetic field around an electron. This field is induced by the coupling of the electron's elementary charge to the gauge field. Such could also be the magnetic monopole character of, say, the electromagnetic field around a Dirac monopole. But why should a particle induce such a field if there exists no magnetic-type elementary charge? There is no reason except for topology! In section 1.3 we briefly mentioned the bundle formalism of gauge theories and we will see that in this formalism topological effects also motivate a third, topological kind of charge, including the quasi-magnetic monopole charge.

In this chapter we define these three kinds of charges and investigate theories of gravity for such charges. It will be very satisfying to recognize the Schwarzschild mass parameter as a *quasi-electric monopole charge of the time translation* and the NUT parameter as a *quasi-magnetic monopole charge of the time translation.* The Kerr parameter is interpreted similarly. These results are in perfect analogy to monopoles in electromagnetism, they shed light on the dimensions of parameters, and they emphasize the analogy between internal and external gauge theories. To obtain these results we formulate Einstein gravity as a gauge theory, i.e. as teleparallelism.

2.1.1 Monopole charges

We start by defining two types of monopole charges. These are properties of the gauge configuration given by the gauge field strength F:

$$\mathcal{E} := \lim_{r\to\infty} \frac{1}{4\pi} \oint\limits_{S^{n-2}(r)} {}^\star F \qquad \textit{quasi-electric monopole charge,} \tag{2.1}$$

$$\mathcal{M} := \lim_{r\to\infty} \frac{1}{4\pi} \oint\limits_{S^{2}(r)} F \qquad \textit{quasi-magnetic monopole charge.} \tag{2.2}$$

The motivation for the definition of $\mathcal{E}$ is obvious from the analogy to Maxwell's inhomogeneous equation. The definition of $\mathcal{M}$ may be motivated by including magnetic charges in Maxwell's theory. Usually this is done by modifying the homogeneous Maxwell equation and introducing a source term on its rhs: $dF = \rho_{\text{mag}}$. But we think it is preferable to interpret $\mathcal{M}$ as the topological invariant associated with the first Chern character class $[F]$ in the second cohomology (see below). With this we don't need to introduce magnetic source terms into the homogeneous Maxwell equation but rather interpret magnetic monopoles as a topological feature – which one may visualize as a Dirac string [13] or rather accept as a feature of a $U(1)$-bundle (see figure 2.1). We choose the nomenclature *quasi-electric* and *-magnetic* to remind us of the analogies with electromagnetism. Since these definitions are general and not restricted to theories of gravitation, we do not choose the names *gravi*-electric and -magnetic.

2.1.2 Topological charges

One principle of topology is comparing manifolds by continuously deforming them. If two manifolds can continuously be deformed into each other, they are said be of the same homotopy type. In topology one is mainly interested in the equivalence classes of manifolds of the same homotopy type. It turns out that there are three important ways of classifying manifolds: First, by identifying all homotopy equivalent manifolds with a set of simplices that are glued together (homology). Second, by considering those forms on the manifold that are closed but not exact (cohomology). In some way (recalling the Stokes theorem) it is not surprising that these two ways of classification are equivalent (de Rham theorem). And third, by considering maps merging a topologically well understood manifold (usually

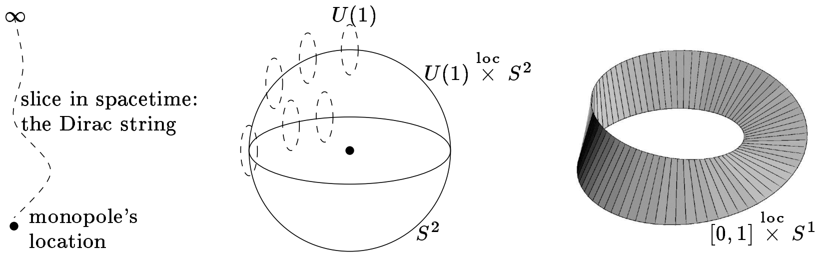


Figure 2.1: The field strength of the Dirac monopole [13] $F = p\, d\Omega = p \sin\theta\, d\theta \wedge d\varphi$ has no global potential A with $F = dA$. Dirac concluded that such a monopole must have a *string* (slice in spacetime) attached to it. If we slice spacetime along the negative z-axis, say, F has a regular potential $A = p\,(1-\cos\theta)\, d\varphi$. Alternatively, electromagnetism may be formulated as a gauge theory on a $U(1)$ bundle over spacetime. Topologically the spacetime *around* the (singular) monopole world path is $(\mathbb{R}^3_{\text{space}} \setminus \{o\}) \times \mathbb{R}_{\text{time}} \sim S^2$, where o denotes the monopole's location. Hence, all field configurations may be classified topologically by investigating $U(1)$ bundles over S^2. It turns out that an integer number (the magnetic charge) classifies all field configurations. The Moebius strip ($[0,1]$ bundle over S^1) allows to visualize a topologically non-trivial bundle.

a r-sphere) into the manifold in question (homotopy). We tried to summarize the essential notions of these theories in appendix C.3.

For us the second way, i.e. considering the cohomology group $H^r(M)$ of r-forms over M that are closed but not exact, is very interesting. The Chern-Weil theorem enables to construct forms out of the gauge field on a fibre bundle that are closed and of which the exactness does *not* depend on the gauge field. Such a form represents *one* element of the cohomology group independent of the gauge field. Hence, this element of the cohomology group indicates an *invariant* (under gauge transformations) topological feature of the bundle. One of these indicators, namely the first Chern character class, may be used to define magnetic monopoles. We will have a closer look on the Chern-Weil theorem below.

But also the third way of classifying the topology of M, i.e. considering the equivalence classes $\pi_r(M)$ of maps merging an r-sphere into M, is very helpful. A theorem proved by Steenrod and Pontrjagin (see, e.g., [2] page 75) states that all G-bundles over the base space S^2 can be classified by $\pi_1(G)$. Since the world-path of a monopole is a singularity, the topology of spacetime in the presence of a monopole is $\mathbb{R}^4 \setminus \{\text{world-path}\} \sim S^2 \times \mathbb{R}^+ \times \mathbb{R} \sim S^2$, i.e. spacetime has the topology of S^2. Hence, all G-bundles over spacetime can be classified by $\pi_1(G)$. In the case of electromagnetism we have $G = U(1)$ and $\pi_1(U(1)) = \mathbb{Z}$, and all gauge field configurations may be characterized by an integer number. This tells us that, in general, there do exist topologically non-trivial gauge configurations in electrodynamics.

Here, we define two topological charges:

$$\mathcal{C}_I := \lim_{r\to\infty} \frac{1}{4\pi} \oint\limits_{S^2(r)\times I} \langle A \wedge F\rangle \qquad \textit{Chern-Simons charge,} \tag{2.3}$$

$$\mathcal{P} := \frac{g^2}{4\pi\hbar^2} \int\limits_{\mathbb{R}^4} \langle F \wedge F\rangle \qquad \textit{Pontrjagin charge.} \tag{2.4}$$

Both charges are fruits of the Chern-Weil theorem which states that these are topological invariants. We remind the reader of the essential ideas of this theorem, for details see [4] or the appendix C.3. First, consider the curvature $F \in \Lambda^2(P, \mathcal{G})$ on a principle bundle P over the base manifold M and formulate polynomials $P(F)$ of this curvature. Then, search for such polynomials that are invariant under the adjoint action of the structure group G, i.e. $\forall g \in G: \; P(\mathrm{Ad}_g F) = P(F)$. Given such an invariant polynomial of r-th order, the Chern-Weil theorem states the following:

(i) $P(F)$ is closed, i.e. $dP(F) = 0$. Hence, we found an element of the $2r$-th cohomology group $[P(F)] \in H^{2r}(M)$. Here, $[P(F)]$ denotes the equivalence class of all $2r$-forms that differ from $P(F)$ only by an exact form. $[P(F)]$ is called *characteristic class*. Note that each monomial in this polynomial is also invariant.
(ii) If we have two curvatures F and F' on the same bundle it follows that $[P(F)] = [P(F')]$. This means that the characteristic class $[P(F)]$ is independent of F and depends only on the topology of the bundle. It is a topological invariant.
(iii) Since $P(F)$ is closed, we find a *local* potential on a subset U of M: $P(F) = dQ\big|_U$. It follows that $[Q]$ is an element of the $(2r{-}1)$-th cohomology $H^{2r-1}(\partial U)$ and is thus a topological invariant of ∂U. Q is called *Chern-Simons form*.

In fact, we find the invariant polynomials (or monomials) $P_1(F) = F$ and $P_2(F) = \langle F \wedge F\rangle$, the first of which is called *1st Chern character term* and the second *1st Pontrjagin term*. We also find the Chern-Simons form $\langle A \wedge F\rangle$ of the 1st Pontrjagin term.

Hence, the 1st Chern character class $[F]$ is an element of the 2nd cohomology. The integration of F over a closed 2-plane S^2, i.e. the quasi-magnetic monopole charge $\mathcal{M}$, thus leads to a number that specifies the cohomology class.

Similarly, the Chern-Simons form $\langle A \wedge F\rangle$ of the 1st Pontrjagin term is an element of the 3rd cohomology and we need a closed 3-plane for integration. In the case of a singular monopole world path in a $U(1)$ bundle, a natural choice for this 3-plane is $S^2 \times U(1)$, with $|U(1)| = \ell_5$. The integration $\mathcal{C}_{U(1)}$ of the Chern-Simons form over this plane thus leads to a finite number classifying the cohomology class. Analogously we have a second choice $I = \mathbb{R}_{\mathrm{time}}$ to form a 3-plane $S^2 \times I$. However, this plane is not compact and will not lead to a finite number. We solve this problem by restricting I to a finite time interval I_T with $|I_T| = T$. Still, the 3-plane $S^2 \times I_T$ is not closed and, strictly speaking, C_{I_T} may not be considered a topological invariant. Thus we have to act with some caution.

The 1st Pontrjagin class $[\langle F \wedge F \rangle]$ is determined by the integration of $\langle F \wedge F \rangle$ over a 4-plane – which we always consider to be spacetime. We will apply this definition in the context of a translational gauge theory, i.e. a geometry with torsion T. Thus, it is very instructive to note that the 'translational Pontrjagin term' $\langle T \wedge T \rangle$ is equivalent to the Nieh-Yan term

$$\mathcal{N} = T^\alpha \wedge T_\alpha - R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta \tag{2.5}$$

in the case of vanishing curvature. The Nieh-Yan term may be produced by splitting the 5-dimensional Pontrjagin term of a deSitter-like $SO(5)$ gauge theory (via some inverse Inou-Wigner contraction) into the 4-dimensional $SO(4)$ Pontrjagin term and the rest. This rest refers to the translations and is, in fact, the Nieh-Yan term (2.5). This was illuminated by Chandia and Zanelli [12].

2.1.3 Elementary charges

For the author, one of the most beautiful things in physics is the success of particle physics in classifying particles with the help of representation theory for groups. This algebraic approach simply postulates that objects in nature must be an element of a representation of some symmetry. We already discussed this approach from a philosophical point of view in section 1.2. Objects (particles or states) that are inseparable are called *elementary.* This notion turns out to coincide with the mathematical notion of *irreducibility.* Both mean *inseparable* without loosing the symmetry (or a faithful representation of it).

With elementary charge we denote those invariants that classify a particle, i.e. the irreducible representation the particle is an element of. Such a classification can be performed by finding all Casimir operators in the group algebra. These are polynomials of the group generators and commute with every group element. Hence, their eigenvalues, when applied on some particle field, are invariant under all symmetry transformations.

The Poincaré group, for example, has the Casimir operators

$$C_1 := P_\alpha P^\alpha \,, \tag{2.6}$$

$$C_2 := W_\alpha W^\alpha \quad \text{with} \quad W_\alpha := \frac{1}{2}\,\epsilon_{\alpha\beta\gamma\delta} L^{\beta\gamma} P^\delta \,. \tag{2.7}$$

Here, the translation operator P^α represents the particle momentum, $L^{\alpha\beta}$ are the generators of Lorentz rotations, and the so-called Pauli-Lubanski vector W^α represents the particle spin. If nature incorporates the Poincaré symmetry, all particles can be classified by eigenvalues of C_1 (mass square) and C_2 (spin square). The classification with respect to their mass is guaranteed by the Dirac equation (for the Dirac spinor representation) or the Klein-Gordon equation (for the scalar representation). All these equations require the dimension $\hbar/\ell$ for the mass parameter. (We take $c = 1$.)

In general, if the Casimir operator C is a polynomial of r-th order of the group generators and if $\mathcal{I}^r$ is an invariant eigenvalue of C, i.e. $(\hbar^r C - \mathcal{I}^r)\,\phi = 0$ for some eigenvector ϕ, then we call $\mathcal{I}$ an elementary charge. If we assume that C is built from generators with dimension $[\lambda_a]$, the dimension of $\mathcal{I}$ is $[\mathcal{I}] = \hbar\,[\lambda_a]$. This leads to the remarkable relation between the dimension of an elementary charge and that of a monopole charge (cf. table 1.3):

$$[\mathcal{I}] = [1/\kappa]\,[\mathcal{E}^a]\,. \tag{2.8}$$

Some further comments on charges in electromagnetism...

Electric charge may be understood as an elementary charge of the single $U(1)$-generator P_5, which is, of course, a Casimir operator. To see this, decompose the $u(1)$-valued connection 1-form into $A = A^5\,P_5$ (with A^5 having the conventional dimension of Weber). The generator P_5 acts trivially on non-charged functions $P_5 \cdot \psi \equiv 0$ but has any charged function as eigenstate $P_5 \cdot \psi = e\,\psi$ with the elementary charge e. The covariant derivative applied on the wave function of an electron, say, reads $D\psi = d\psi + A\cdot\psi = d\psi + A^5\,P_5\cdot\psi = d\psi + eA^5\psi$, as we are used to write it. This coupling of the elementary charge to the gauge field induces the electric monopole character of the electromagnetic field. Hence, the electric charge density ϱ may be understood as elementary charge density. A magnetic monopole character, though, cannot be induced be an elementary charge since there exists no second, magnetic-type Casimir operator. Hence, some ϱ_{mag} on the rhs of the inhomogeneous Maxwell equation may merely be understood as a density of *topological defects*, but not as elementary charge density.

... and on mass as an elementary charge

First, the dimension of the mass parameter $[m] = \hbar/\ell$ may be called *phase*/2π *per length.* In fact, the most obvious argument for this interpretation is the point particle action $\int m\,ds$. In this picture, if you identify a world path with a strap, then mass is the twist of this strap per length. Also note that $\lambda_c = \hbar/m$ is the Compton *wave* length of the particle. Second, since in 5D Kaluza-Klein space the electric charge q is just as well an eigenvalue of the Casimir operator of the *translation* along the 5th dimension, electric charge is very similar to mass. Just as mass measures the *horizontal* (spacetime) momentum, the electric charge measures a *vertical* (fibre) momentum. In fact, Bleecker [1] defined electric charge as the 'vertical velocity' of a point particle path on a $U(1)$-bundle. Third, Rosen [31] introduced a *massive* Klein-Gordon field by multiplying a *phase* $\exp(-i\, m_0\, t)$ to a real scalar field ϕ. Unfortunately, Rosen does not motivate this in a very detailed manner. Very interestingly, in the case of flat spacetime, the parameter m_0 introduced by attaching this phase $\exp(-i\, m_0\, t)$ to the scalar field ϕ *cancels* with the parameter m_1 introduced by adding a square term $(m_1)^2(\phi \wedge {}^\star\phi)$ to the lagrangian.

2.2 Translational monopole charges in gauge theories of gravity

We can now apply the charge definitions to analyze standard solutions of gauge theories of gravity for monopole charges. First, we concentrate on a subclass of the Plebanski-Demianski class of solutions including the Kerr-Newman and Taub-NUT solutions. For the monopole analysis we formulate them as a solution of a translational gauge theory of gravity, namely teleparallelism, and find quasi-electric and quasi-magnetic monopoles in the gauge of some translations, indeed. Later, we also investigate two solutions of the Poincaré gauge theory.

Before we start we should point out that the following would hardly have been possible without the use of the computer algebra system Reduce and its supplementary package Excalc. The calculations for the monopole analysis are rather straightforward but very extensive. In appendix D we display the Reduce files. These are very short and should easily be readable because they rely on the library `magtools` that the author wrote in order to implement basic routines in the context of gauge theory of gravity in a standard way.

2.2.1 The Kerr-Newman solution

In polar coordinates (t, r, θ, φ), the Kerr-Newman metric with mass parameter m, Kerr parameter j, electric charge q, and magnetic charge p reads

$$g = \vartheta^{\hat{0}} \otimes \vartheta^{\hat{0}} - \vartheta^{\hat{1}} \otimes \vartheta^{\hat{1}} - \vartheta^{\hat{2}} \otimes \vartheta^{\hat{2}} - \vartheta^{\hat{3}} \otimes \vartheta^{\hat{3}} \, , \tag{2.9}$$

$$\vartheta^{\hat{0}} = \frac{\mathcal{Q}}{\Delta} \, d\tau \, , \quad \vartheta^{\hat{1}} = \frac{\Delta}{\mathcal{Q}} \, dr \, , \quad \vartheta^{\hat{2}} = \frac{\Delta}{\mathcal{P}} \sin\theta \, d\theta \, , \quad \vartheta^{\hat{3}} = \frac{\mathcal{P}}{\Delta} \, d\sigma \, , \tag{2.10}$$

$$d\tau = dt - j \sin^2\theta \, d\varphi \, , \quad d\sigma = (r^2 + j^2) \, d\varphi - j \, dt \, , \tag{2.11}$$

$$\mathcal{Q}^2 = r^2 - 2mr + j^2 + \frac{1}{4}(q^2 + p^2) \, , \quad \mathcal{P} = \sin\theta \, , \quad \Delta^2 = r^2 + j^2 \cos^2\theta \, . \tag{2.12}$$

This notation might confuse at first. It is the direct analogue of the notation Plebanski and Demianski used in their paper [30]. It has a clear structure and can easily be modified into other solutions of the Plebanski-Demianski class. The metric solves the coupled Einstein-Maxwell equations if we choose the electromagnetic potential

$$A = \frac{1}{\Delta^2}\big(q \, r \, d\tau + p \cos\theta \, d\sigma\big) \, . \tag{2.13}$$

This potential is the analog of the potential $A = q/r \, dt + p \cos\theta \, d\varphi$ of an electric and magnetic charge in flat spacetime. For the monopole analysis, we translate this solution into a 5D Kaluza-Klein-type teleparallelism as described in section 1.5.2. This simply means that we add a 5th dimension that represents the electromagnetic part of the theory:

$$g = \vartheta^{\hat{0}} \otimes \vartheta^{\hat{0}} - \vartheta^{\hat{1}} \otimes \vartheta^{\hat{1}} - \vartheta^{\hat{2}} \otimes \vartheta^{\hat{2}} - \vartheta^{\hat{3}} \otimes \vartheta^{\hat{3}} - \vartheta^{\hat{5}} \otimes \vartheta^{\hat{5}} \, , \tag{2.14}$$

$$\vartheta^{\hat{5}} = dx^5 + \frac{1}{\Delta^2}\big(q \, r \, d\tau + p \cos\theta \, d\sigma\big) \, . \tag{2.15}$$

The 5th covector $\vartheta^{\hat{5}}$ represents the gauge of the 5th translation, i.e. the electromagnetic gauge potential. The field strength of this gauge theory is the torsion $T^\alpha = d\vartheta^a$. The configuration solves the vacuum field equation $dH^a = 0$ of the teleparallelism theory with the excitation H^α given in (1.62).

The following charges for this gauge configuration are calculated by the file `kerrnut.exi` in appendix D with parameters `(m,j,q,p)`:

$$\mathcal{E} = -m \, \partial_t - j \frac{\pi}{4} \, \partial_\varphi + q \, \partial_5 \, , \quad \mathcal{M} = -p \, \partial_5 \, , \quad \mathcal{C}_{U(1)} = -p \, \ell_5 \, , \quad \mathcal{C}_{I_T} = 0 \, , \quad \mathcal{P} = 0 \, . \tag{2.16}$$

Consider $\mathcal{E}$ and note that we have a quasi-electric monopole charge $\mathcal{E}^t = -m$ in the time translation, a quasi-electric monopole charge $\mathcal{E}^\varphi = -j\frac{\pi}{4}$ in the translation along ∂_φ

(which is actually a rotation and the charge represents an angular momentum)[1], and a (quasi-)electric monopole charge $\mathcal{E}^5 = q$ in the translation along ∂_5 (i.e. the $U(1)$ gauge of electrodynamics). In this solution all Killing vectors carry quasi-electric monopole charges. In fact, it seems quite plausible that the elementary charges of the three Casimir operators (momentum square, Pauli-Lubanski square, and the 5th translation) are the sources of the quasi-electric monopole charges of the Killing vectors that correspond to these Casimirs in a stationary geometry. As we are interested in dimensions, we find that the mass parameter has dimension $[m] = \ell$, the angular momentum per mass unit has dimension $[j] = 1$, and, if we measure the length along the 5th dimension in units of ℓ_5, the electric charge has dimension $[q] = \ell_5$. In the previous dimensional discussion of electrodynamics, we defined $1/\ell_5 = e/\hbar$ and $[1/\kappa] = e^2/\hbar = \hbar/\ell_5^2$. Hence, our results are consistent with eq. (2.8): The dimension of the elementary charge $[\mathcal{I}] = e = \hbar/\ell_5$ is equal to the coupling constant $[1/\kappa]$ times the dimension of the quasi-electric monopole charge $[\mathcal{E}^5] = [q] = \ell_5$. The same holds for the mass.

Considering $\mathcal{M}$ we are not surprised that $\mathcal{M}^5 = -p$ is a (quasi-)magnetic monopole charge of the 5th translation. The non-trivial Chern-Simons form $\mathcal{C}_{U(1)}$ confirms the topological feature of magnetic monopoles in the $U(1)$-bundle.

A degenerate case of the Kerr-Newman solution

We add this brief remark here to present a confusingly simple case of the Kerr-Newman solution. The upper configuration is a solution of the Kaluza-Klein model. Hence, as pointed out before, it does not need to solve the $\hat{5}\hat{5}$-component of the Einstein or teleparallel field equation. In fact, in the case $j = 0$ (i.e. the Reissner-Nordström solution) we are left with the following external energy-momentum 4-form on the right hand side of the 5-dimensional field equation:

$$\Sigma^{\hat{0}..\hat{3}} = 0\,, \quad \Sigma^{\hat{5}} = \frac{3(p^2 - q^2)}{2\kappa r^4}\,\eta\,. \tag{2.17}$$

Hence, it is easy to find a solution for the full 5-dimensional Einstein or teleparallel gravity that also solves the $\hat{5}\hat{5}$-component by simply requiring $p = q$. Furthermore, we get a

[1] Usually, one associates a *gravi-magnetic* or *gravito-magnetic* effect with the gravitational field of the Kerr solution. This is sensible since the rotating mass produces a field that is in analogy to the magnetic field produced by rotating electrons. However, rotating mass is not an analogue to a magnetic *monopole*. Instead, our calculation definitely proves that it is rather in analogy to an electric monopole – but with respect to the gauge of translations along the Killing vector ∂_φ.

surprisingly simple and unusual solution if we also set $q = \sqrt{2}\, m$. The coframe then reads

$$\vartheta^{\hat{0}} = \left|1 - \frac{m}{r}\right| dt \,, \quad \vartheta^{\hat{1}} = \frac{1}{\left|1 - \frac{m}{r}\right|} dr \,, \quad \vartheta^{\hat{2}} = r\, d\theta \,, \quad \vartheta^{\hat{3}} = r\, \sin\theta d\varphi \,. \tag{2.18}$$

The solution is well known and also presented in [16], e.g.

2.2.2 The Taub-NUT solution

Let us turn to the Taub-NUT solution with mass parameter m, NUT parameter n, and electric charge q. Within the previous notation, i.e. with the coframe and metric defined in (2.14, 2.10, 2.15), the solution reads

$$d\tau = dt - 2n \cos\theta\, d\varphi \,, \quad d\sigma = (r^2 + n^2)\, d\varphi \,, \tag{2.19}$$
$$p = 0 \,, \quad \mathcal{Q}^2 = r^2 - 2mr - n^2 + q^2/4 \,, \quad \mathcal{P} = \sin\theta \,, \quad \Delta^2 = r^2 + n^2 \,. \tag{2.20}$$

The result of the monopole analysis has been calculated with the program `kerrnut.exi` in appendix D with parameters `(m,n,q)`:

$$\mathcal{E} = -m\,\partial_t + q\,\partial_5 \,, \quad \mathcal{M} = -2n\,\partial_t \,, \quad \mathcal{C}_{U(1)} = 0 \,, \quad \mathcal{C}_{I_T} = -2n\,T \,, \quad \mathcal{P} = 4n - \frac{q^2}{2n} \,. \tag{2.21}$$

This clearly presents the NUT parameter n as a quasi-magnetic monopole charge of the time translation. Table 2.1 gives another illustration of these results.

2.2.3 The McCrea-NUT solution

The charged McCrea-NUT solution of the Poincare gauge theory [25] solves the field equations (1.53) and (1.54) of the van der Heyde lagrangian with excitations

$$H_{\text{vdH}}{}^{\alpha} = {}^{\star}\overset{1}{T}{}^{\alpha} - 2{}^{\star}\overset{2}{T}{}^{\alpha} + {}^{\star}\overset{3}{T}{}^{\alpha} \,,$$
$$H_{\text{vdH}\,\alpha}{}^{\beta} = {}^{\star}R_{\alpha}{}^{\beta} \,. \tag{2.22}$$

electric monopole	Schwarzschild solution
$A = -\frac{q}{r}\,dt$ $F = -\frac{q}{r^2}\,dt \wedge dr$	$Y^{\hat{0}} = \vartheta^{\hat{0}} - dt = \left(\sqrt{1-\frac{2m}{r}} - 1\right) dt \longrightarrow -\frac{m}{r}\,dt$ $T^{\hat{0}} \longrightarrow -\frac{m}{r^2}\,dt \wedge dr$
magnetic monopole	Taub-NUT solution
$A = p(1-\cos\theta)\,d\varphi$ $F = dA = p\,d\Omega$	$Y^{\hat{0}} = \vartheta^{\hat{0}} - dt \longrightarrow 2n(1-\cos\theta)\,d\varphi$ $T^{\hat{0}} \longrightarrow 2n\,d\Omega$

Table 2.1: The table compares the electric monopole with the Schwarzschild solution and the Dirac monopole with the Taub-NUT solution. The gravitational solutions are presented in a teleparallel formalism. Arrows $\longrightarrow$ mean the limit $r \to \infty$. The analogies between the electro-magnetic field strength F and the field strength of time translation $T^{\hat{0}}$ confirm our interpretation of the mass parameter m and the NUT parameter n. The identification of $\vartheta^{\hat{0}} - dt$ with the gauge potential of time translation $Y^{\hat{0}}$ takes the soldering (1.29) into account.

The solution is parameterized by m, n, and q. Although it is charged we do not consider the Kaluza-Klein extension. The corresponding Reduce file is `mccrea.exi`.

$$
\begin{aligned}
&g = \vartheta^{\hat{0}} \otimes \vartheta^{\hat{0}} - \vartheta^{\hat{1}} \otimes \vartheta^{\hat{1}} - \vartheta^{\hat{2}} \otimes \vartheta^{\hat{2}} - \vartheta^{\hat{3}} \otimes \vartheta^{\hat{3}} \,, && (2.23)\\
&\vartheta^{\hat{0}} = (\psi/4 + 1/2)(dt + 2n\cos\theta\,d\varphi) + (1/2 - 1/\psi)\,dr \,, \\
&\vartheta^{\hat{1}} = (\psi/4 - 1/2)(dt + 2n\cos\theta\,d\varphi) + (1/2 + 1/\psi)\,dr \,, \\
&\vartheta^{\hat{2}} = \Delta\,d\theta \,, \quad \vartheta^{\hat{3}} = \Delta\sin\theta\,d\varphi \\
&\qquad \psi := \frac{-8n^2(mr-q^2) - \Delta^4}{2n^2\Delta^2} \,, \quad \Delta^2 := r^2 + n^2 \,, && (2.24)
\end{aligned}
$$

$$
\begin{aligned}
&T^{\hat{0}} := f\ \vartheta^{\hat{0}\hat{1}} - 2c_3\ \vartheta^{\hat{2}\hat{3}} \,, \\
&T^{\hat{1}} := -h\ \vartheta^{\hat{0}\hat{1}} + 2c_4\ \vartheta^{\hat{2}\hat{3}} \,, \\
&T^{\hat{2}} := -k\ \vartheta^{\hat{0}\hat{2}} - (c_3 - c_1)\ \vartheta^{\hat{0}\hat{3}} + (c_4 - c_2)\ \vartheta^{\hat{3}\hat{1}} + g\ \vartheta^{\hat{1}\hat{2}} \,, \\
&T^{\hat{3}} := (c_3 - c_1)\ \vartheta^{\hat{0}\hat{2}} - k\ \vartheta^{\hat{0}\hat{3}} - g\ \vartheta^{\hat{3}\hat{1}} + (c_4 - c_2)\ \vartheta^{\hat{1}\hat{2}} \\
&\qquad c_3 := \frac{-n(mr-q^2)}{\Delta^4} \,, \quad c_1 := 2c_3 \,, \quad c_4 := -c_3 \,, \quad c_2 := 2c_4 \,, \\
&\qquad f := \frac{-r(mr-2q^2) + mn^2}{\Delta^4} \,, \quad g := \frac{-r(mr-q^2)}{\Delta^4} \,, \quad h := -f \,, \quad k := g \,, && (2.25)
\end{aligned}
$$

$$
\mathcal{E} = -m(e_{\hat{0}} + e_{\hat{1}}) \,, \quad \mathcal{M} = 0 \,, \quad \mathcal{C}_{U(1)} = 0 \,, \quad \mathcal{C}_{I_T} = 0 \,, \quad \mathcal{P} = 0 \,. \qquad (2.26)
$$

We do not find any quasi-magnetic or topological charges in this solution. The quasi-electric monopole only confirms the massive nature of this solution. This is similar to the following solution:

2.2.4 The Baekler-Kerr-Newman solution

Finally we want to investigate a solution of the Poincaré gauge theory found by Baekler et al. [11]. The solution was found by a double-duality ansatz (see e.g. [10]). It represents the *exterior gravitational field of a charged spinning source in the Poincaré gauge theory – a Kerr-Newman metric with dynamical torsion.* It also solves the field equations (1.53) and (1.54) of the van der Heyde lagrangian. The solution is parameterized by m, n, j, q. The Reduce file is `baekler.exi`.

$$
\begin{aligned}
& g = \vartheta^{\hat{0}} \otimes \vartheta^{\hat{0}} - \vartheta^{\hat{1}} \otimes \vartheta^{\hat{1}} - \vartheta^{\hat{2}} \otimes \vartheta^{\hat{2}} - \vartheta^{\hat{3}} \otimes \vartheta^{\hat{3}} \ , \\
& \vartheta^{\hat{0}} = \frac{\mathcal{Q}}{\Delta}\, d\tau \ , \quad \vartheta^{\hat{1}} = \frac{\Delta}{\mathcal{Q}}\, dr \ , \quad \vartheta^{\hat{2}} = \frac{\Delta}{\mathcal{P}} \sin\theta\, d\theta \ , \quad \vartheta^{\hat{3}} = \frac{\mathcal{P}}{\Delta}\, d\sigma \ , \\
& \qquad d\tau = dt + (j \sin^2\theta + 2n\cos\theta)\, d\varphi \ , \quad d\sigma = (r^2 + j^2 + n^2)\, d\varphi + j\, dt \ , \\
& \qquad \mathcal{P}^2 := \sin^2\theta \big[1 - \frac{1}{3}\Lambda j \cos\theta (j\cos\theta - 4n)\big] \ , \\
& \qquad \Delta^2 := r^2 + (j\cos\theta - n)^2 \ , \\
& \qquad \mathcal{Q}^2 := -2(mr - q^2) + (r^2 + j^2 - n^2) + \frac{1}{3}\Lambda \left[r^2(r^2 + j^2 + 6n^2) + 3n^2(j^2 - n^2)\right] \ , \\
& \qquad \Lambda := \frac{3}{j^2 - 4n^2} \ ,
\end{aligned}
\tag{2.27}
$$

$$
\begin{aligned}
& T^{\hat{0}} := -V_1\, \vartheta^{\hat{0}\hat{1}} + V_2(\vartheta^{\hat{0}\hat{2}} - \vartheta^{\hat{1}\hat{2}}) + V_3(\vartheta^{\hat{0}\hat{3}} - \vartheta^{\hat{1}\hat{3}}) - 2V_4\, \vartheta^{\hat{2}\hat{3}} \ , \\
& T^{\hat{1}} := T^{0} \ , \\
& T^{\hat{2}} := \ \ V_5(\vartheta^{\hat{0}\hat{2}} - \vartheta^{\hat{1}\hat{2}}) + V_4(\vartheta^{\hat{0}\hat{3}} - \vartheta^{\hat{1}\hat{3}}) \ , \\
& T^{\hat{3}} := -V_4(\vartheta^{\hat{0}\hat{2}} - \vartheta^{\hat{1}\hat{2}}) + V_5(\vartheta^{\hat{0}\hat{3}} - \vartheta^{\hat{1}\hat{3}}) \ , \\
& \qquad V_1 := \frac{1}{\mathcal{Q}\Delta^3}\ \left[(mr - 2q^2)r - m(j\cos\theta - n)^2\right] \ , \\
& \qquad V_4 := \frac{1}{\mathcal{Q}\Delta^3}\,(mr - q^2)(j\cos\theta - n) \ , \quad V_5 := \frac{1}{\mathcal{Q}\Delta^3}\,(mr - q^2)r \ , \\
& \qquad V_2 := -\frac{\mathcal{P}}{\mathcal{Q}}\, j\, V_4 \ , \quad V_3 := \frac{\mathcal{P}}{\mathcal{Q}}\, j\, V_5 \ ,
\end{aligned}
\tag{2.28}
$$

$$
\mathcal{E} = 0 \ , \quad \mathcal{M} = 0 \ , \quad \mathcal{C}_{U(1)} = 0 \ , \quad \mathcal{C}_{I_T} = 0 \ , \quad \mathcal{P} = 0 \ . \tag{2.29}
$$

This negative result might surprise at first. However, note the following limits for $r \to \infty$:

$$Q \to \sqrt{\frac{\Lambda}{3}}\, r^2 \,, \quad V_1 \to \sqrt{\frac{3}{\Lambda}}\, \frac{m}{r^3} \,, \quad V_4 \to \sqrt{\frac{3}{\Lambda}}\, \frac{m(j\cos\theta - n)}{r^4} \,,$$
$$T^{\hat{0}} \to -\sqrt{\frac{3}{\Lambda}}\, \frac{m}{r^3}\, dt \wedge dr - 2\sqrt{\frac{3}{\Lambda}}\, \frac{m(j\cos\theta - n)}{r^4} \sin\theta \, d\theta \wedge d\varphi \,. \tag{2.30}$$

Obviously, those solutions of the Poincaré gauge theory we analyzed do not express their parameters as translational charges, i.e. charges of *torsion*. We think that the reason for this is the mechanism that was used to generate these solutions – the double duality ansatz (see [10]). It seems that in this ansatz the curvature and not torsion carries the characters of the solution as $r \to \infty$, as it is in the Riemannian formulation of Einstein gravity, too.

2.3 Relating to other formalisms

In this short section we will display the relation of our analysis to more conventional ones. Lynden-Bell et al. [21], e.g., wrote a detailed review on monopoles in gravity and also discussed the magnetic nature of NUT-space. Their considerations are based on the following definitions of the gravo-electric and -magnetic fields. They point out that a time-like Killing vector is necessary for this definition and hence they consider the general *stationary* metric (cf. [21] eq (3.1))

$$g = f^2\,(dx^0 - A_i dx^i)^2 - \gamma_{ij}\, dx^i\, dx^j \,, \tag{2.31}$$

where $i = 1, 2, 3$ and A_i and γ_{ij} are arbitrary. Motivated by the expression of the force on a test particle with rest mass m_0 and velocity $\vec{v}$ in this geometry (cf. [21] (3.2))

$$\vec{f} = \frac{m_0}{\sqrt{1 - v^2/c^2}} \left[\left(-\frac{1}{f}\, \vec{\nabla} f \right) + \frac{v}{c} \times \left(f \operatorname{curl} \vec{A} \right) \right] \,, \tag{2.32}$$

and its formal analogy to the electromagnetic Lorentz force, they define the gravo-electric and -magnetic fields as (cf. [21] (3.3,3.4))

$$\vec{E} := -\frac{1}{f}\, \vec{\nabla} f \,, \tag{2.33}$$
$$\vec{B} := \operatorname{curl} \vec{A} \,. \tag{2.34}$$

We can now give another interpretation of these definitions by reproducing them in our teleparallel formalism. The metric (2.31) is replaced by the coframe with

$$\vartheta^{\hat{0}} = f\, (dx^0 - A) \,, \tag{2.35}$$

together with three spatial covectors ϑ^i that are of no further interest. We introduced the space-like 1-form $A = A_i\, dx^i$. Since in table 2.1 we notice a close relation between Newton's force and the time component of torsion $T^{\hat{0}}$, we calculate

$$\begin{aligned} T^{\hat{0}} &= d\vartheta^{\hat{0}} = df \wedge (dx^0 - A) - f\, dA \\ &= \frac{1}{f}\, df \wedge \vartheta^{\hat{0}} - f\, dA\,. \end{aligned} \tag{2.36}$$

Following the conventional space-time decomposition of the electromagnetic force we split this field strength of time translation into an electric and magnetic part:

$$T^{\hat{0}} = -\big(E \wedge \vartheta^{\hat{0}} + B\big)\,, \tag{2.37}$$

$$E := -\frac{1}{f}\, df\,, \tag{2.38}$$

$$B := f\, dA\,. \tag{2.39}$$

Thereby we reproduced the definitions (2.33,2.34) up to the factor f in B. However, looking at the force (2.32) it seems more consistent to include this factor f in B in order to arrive at the conventional expression for the Lorentz force. We conclude that the conventional formalism presented by Lynden-Bell et al. is equivalent to our investigation in monopoles in the *time-component* of torsion $T^{\hat{0}}$. However, their formalism is non-covariant at its very basis, it is insufficient to discuss monopole charges in other translations (e.g. the Kerr parameter as quasi-electric monopole charge in the translation along ∂_5), and it does not allow to identify quasi-magnetic charges with Chern-Simons charges in the way we did. Finally, we cite the interesting statement of Rindler [7] section 8.12 according to which the minus sign in (2.37) – which is the only difference to the electromagnetic paradigm – is due to the *attractive* nature of the gravitational force.

2.4 Summary and discussion

This chapter clearly divides into two parts, the first defining some notions of charges, and the second calculating them for different solutions of gauge theories of gravity. Besides merely defining (2.1-2.4) in section 2.1, we tried to emphasize and clarify the topological background. In particular, we stressed that a quasi-magnetic charge should be motivated topologically. Equation (2.8) relates the dimension of elementary charges to that of monopole charges via the coupling constant and, perhaps, the *comments on mass as elementary charge* inspired some intuition for what mass is. The major result of section 2.2 are the charges (2.16) and (2.21) of the Kerr-Newman and Taub-NUT solution in the teleparallel formulation, respectively. Table 2.2 summarizes the interpretation of these charges.

Casimir operators (∼ elementary charges)	Killing vectors	quasi-electric monopole charges	quasi-magnetic monopole charges
$C_1 := P_\mu P^\mu$	∂_t	m	n
$C_2 := W_\mu W^\mu$	∂_φ	j	$(a?)$
P_5	(∂_5)	q	p

Table 2.2: The correspondence between Casimir operators, Killing vectors, and monopole charges in the Plebanski-Demianski class of solutions. The three columns to the right refer to a *stationary* (and spherically symmetric) geometry. We have the Schwarzschild mass parameter m, Taub-NUT parameter n, Kerr parameter j, acceleration parameter a, electric charge q, and magnetic charge p.

All this has only been possible because of the gauge theoretical formulation of gravity and stresses the analogies between internal and external gauge theories. As a byproduct we developed the Reduce library `magtools` that implements basic routines in the context of MAG but also procedures that calculate the monopole and topological charges. One can find it in appendix D. Finally, we want to stress the following points:

(1) As we discussed in section 2.3, the (gravo-) electric and magnetic nature of the Schwarzschild and Taub-NUT solution, respectively, can also be pointed out in the Riemannian formulation of gravity. However, in the teleparallel formalism we arrived to recover the Schwarzschild mass parameter and the NUT-parameter as monopole charges *of the time-translation*. First, this explains why Lynden-Bell et al. need to assume a time-like Killing vector for their definitions of gravo-electric and -magnetic fields, and second, this uncovers the analogy between those charges and charges of other translations, namely those along ∂_φ and ∂_5. Furthermore, our definitions (2.1,2.2) have the advantage to be covariant.

(2) Teleparallelism formulates Einstein gravity as a gauge theory and, in table 2.1, we found a remarkable analogy between teleparallelism and Newtonian gravity: The time component of torsion is in close analogy to Newton's force (closer than the Riemannian curvature is). Accordingly, the time component of the translational potential $\vartheta^{\hat{0}} - dt$ is close to the Newtonian potential.

(3) We proved that in the Plebanski-Demianski class of solutions [30] (when reformulated as teleparallel solutions) the five parameters m, n, q, p, and j may be related to monopole charges. Unfortunately, we could not confirm the same for the acceleration parameter a. The reason might be the topologically non-trivial coordinate transformation eq (4.4) in [30]. However, for consistency we may expect that a relates to a quasi-magnetic charge of the translation along ∂_φ. Assuming this, we agree with Plebanski and Demianski on their ordering of the parameters: The six parameters should be ordered as three pairs (m,

n), (j, a), and (q, p) each pair of which belongs to the time translation, the translation along ∂_φ, and the $U(1)$-translation, respectively. In each pair the first parameter denotes the quasi-electric charge and the second parameter the quasi-magnetic charge of these translations.

Chapter 3

A numeric solution of MAG

3.1 Introduction

The motivation for this chapter has its origin in an observation we made in section 1.4.4 where, induced by a metric g, we split the linear symmetry into an antisymmetric and symmetric part. We found that the symmetric curvature $\overset{\circ}{R} = \frac{1}{2}(\hat{D}Q - \hat{\Gamma} dg)$ (1.41) includes the differential $\hat{D}Q$ of the nonmetricity $Q := Dg$. Hence, a lagrangian that includes square terms like $\overset{\circ}{R}{}^2$ and Q^2 might produce an effective theory that treats Q as a massive, propagating field – such as the Proca field is. These ideas are a vast simplification of the exact considerations Obukhov et al. made in [27]. Also, one has to restrict the lagrangian such that only a square term of the trace of R is left which represents the dilation part of the original $GL(n)$ symmetry. All other curvature square terms (that represent *strong gravity*) are set to zero. In detail, given the curvature $R_\alpha{}^\beta$ and its 6+5 irreducible pieces ${}^{(I)}\hat{R}_\alpha{}^\beta$ and ${}^{(I)}\overset{\circ}{R}_\alpha{}^\beta$, the torsion T_α and its 3 irreducible pieces ${}^{(I)}T_\alpha$, and the nonmetricity $Q_{\alpha\beta}$ and its 4 irreducible pieces ${}^{(I)}Q_{\alpha\beta}$ (see [17] appendix B), we may formulate the general

MAG lagrangian as

$$
\begin{array}{rll}
\mathcal{L}_{\rm MAG} = & \frac{1}{2\kappa}\Big[- \; a_0 R^{\alpha\beta}\wedge\eta_{\alpha\beta} - 2\lambda\eta & \text{à la Hilbert-Einstein} \\
& + \; T^{\alpha}\wedge{}^{\star}(a_{I=1..3}{}^{(I)}T_{\alpha}) & \text{quadratic torsion} \\
& + \; Q_{\alpha\beta}\wedge{}^{\star}(b_{I=1..4}{}^{(I)}Q^{\alpha\beta}) & \text{quadratic nonmetricity} \\
& + \; b_5\,({}^{(3)}Q_{\alpha\gamma}\wedge\vartheta^{\alpha})\wedge{}^{\star}({}^{(4)}Q^{\beta\gamma}\wedge\vartheta_{\beta}) & \text{quad. nonm. mixed with } \vartheta^{\alpha} \\
& + \; 2(c_{I=2..4}\,{}^{(I)}Q_{\alpha\beta})\wedge\vartheta^{\alpha}\wedge{}^{\star}T^{\beta}\Big] & \text{cross terms nonm./torsion} \\
- & \frac{1}{2\varrho}R^{\alpha\beta}\wedge{}^{\star}\Big[(w_{I=1..6}{}^{(I)}\hat{R}_{\alpha\beta}) + (z_{I=1..5}{}^{(I)}\overset{\circ}{R}_{\alpha\beta}) & \text{quadratic curvature} \\
& + w_7\,\vartheta_{\alpha}\wedge(e_{\gamma}\rfloor{}^{(5)}\hat{R}^{\gamma}{}_{\beta}) + z_6\,\vartheta_{\gamma}\wedge(e_{\alpha}\rfloor{}^{(2)}\overset{\circ}{R}^{\gamma}{}_{\beta}) & \\
& + z_{I=7..9}\,\vartheta_{\alpha}\wedge(e_{\gamma}\rfloor{}^{(I-4)}\overset{\circ}{R}^{\gamma}{}_{\beta})\Big]\,. & \Big\}\text{mixed with } \vartheta^{\alpha}
\end{array}
\tag{3.1}
$$

This lagrangian and the presently known solutions have been reviewed in [16]. We have the *weak* and *strong* coupling constants $1/\kappa$ and $1/\varrho$, the cosmological constant λ, and the 28 parameters $a_{I=0..3}$, $b_{I=1..5}$, $c_{I=2..4}$, $w_{I=1..7}$, and $z_{I=1..9}$. We always sum over I, as we do with other indices. Note that the weak coupling constant has length dimension $[1/\kappa] = 1/\ell^2$ because it multiplies to a torsion square, torsion being the field strength of translation generators with dimension $1/\ell$. (See the discussion in section 1.6.) The strong coupling constant, though, has no length dimension. The work presented here is only concerned with the special case

$$
\varrho = 1\,, \quad w_{I=1..7} = 0\,, \quad z_{I=1..3} = z_{I=5..9} = 0\,, \quad z_4 \neq 0\,. \tag{3.2}
$$

This means that we consider a general *weak lagrangian* (with weak coupling constant) but only a very constrained *strong lagrangian* (of curvature squares) allowing only for a square of the dilation curvature ${}^{(4)}\overset{\circ}{R}_{\alpha\beta} := \frac{1}{n}\,g_{\alpha\beta}\,R_{\gamma}{}^{\gamma}$. Qualitatively, the lagrangian with these constraints may be displayed as

$$
\mathcal{L} \;\sim\; \lambda \;+\; R \;+\; \big(T+Q\big)^2 \;+\; \big({}^{(4)}\overset{\circ}{R}\big)^2\,. \tag{3.3}
$$

Here, R, T, and Q denote just some terms linear in the curvature, torsion, and nonmetricity, respectively. On this qualitative level, the result of Obukhov et al. [27] is the following: Effectively, the curvature R may be considered Riemannian, T and Q may be replaced by a 1-form ϕ, i.e. $T+Q\sim\phi$, and ${}^{(4)}\overset{\circ}{R}$ is similar to $d\phi$. Hence, (3.3) reads generically

$$
\mathcal{L} \;\sim\; \lambda \;+\; R_{\rm riem} \;+\; \phi^2 \;+\; (d\phi)^2\,, \tag{3.4}
$$

which describes an Einstein spacetime with a massive 1-form field ϕ, i.e. a Proca field.

We will review the results of [27] in detail. First, one considers *a special case* of the MAG lagrangian (3.1) with constraint (3.2) by specifying the remaining parameters λ, $a_{I=0..3}$, $b_{I=1..5}$, $c_{I=2..4}$, and z_4. This choice is done in [27] eq (4.1) and turns out to effectively produce a purely Riemannian Hilbert-Einstein lagrangian, cf. [27] (4.6). This allows to introduce a new variable, the effective Riemannian curvature. Then, having investigated this special lagrangian, they generalize it again by adding a general lagrangian (constrained by (3.2)) to it, [27] (5.1-5.5). With the aid of the effective Riemannian curvature, the field equation FIRST (the variation with respect to the coframe (1.53)) reads like an Einstein equation with an energy-momentum source that depends on torsion and nonmetricity, [27] (5.11). The field equation SECOND (the variation with respect to the linear connection (1.54)) becomes a system of differential equations for torsion and nonmetricity alone. In the vacuum case, where the energy-momentum Σ_α and the hypermomentum $\Delta_\alpha{}^\beta$ of matter vanish, these differential equations (i.e. SECOND) reduce to

$$
\begin{aligned}
&{}^{(1)}T_\alpha = 0\,, \quad {}^{(2)}T_\alpha = \frac{k_2}{3}\,\vartheta_\alpha \wedge \phi\,, \quad {}^{(3)}T_\alpha = 0\,, && \text{cf. [27] (6.2,6.6+2.3,5.20)}\\
&{}^{(1)}Q_{\alpha\beta} = 0\,, \quad {}^{(2)}Q_{\alpha\beta} = 0\,, && \text{cf. [27] (5.27,6.3)}\\
&{}^{(3)}Q_{\alpha\beta} = \frac{4}{9}k_1\left(\vartheta_{(\alpha}e_{\beta)}\rfloor\phi - \frac{1}{4}g_{\alpha\beta}\,\phi\right)\,, \quad {}^{(4)}Q_{\alpha\beta} = k_0\,g_{\alpha\beta}\,\phi\,, && \text{cf. [27] (6.5+2.7,2.8)}\\
&d^\star d\phi + m^2\phi = 0\,. && \text{cf. [27] (6.7)}
\end{aligned}
\tag{3.5}
$$

Here, the new 1-form ϕ determines the torsion and nonmetricity completely and needs to solve the Proca equation. The four constants m, k_0, k_1, and k_2 uniquely depend on the parameters of the MAG lagrangian via [27] (6.8, 6.4, 5.3-5.5). We summarize

$$
\begin{aligned}
k_0 &= 4(a_2 - 2a_0)(b_3 + a_0/8) - 3(c_3 + a_0)^2\,,\\
k_1 &= 9/2(a_2 - 2a_0)(b_5 - a_0) - 9(c_3 + a_0)(c_4 + a_0)\,,\\
k_2 &= 12(b_3 + a_0/8)(c_4 + a_0) - 9/2(b_5 - a_0)(c_3 + a_0)\,,\\
m^2 &= \frac{1}{z_4\kappa}\left(-4b_4 + \frac{3}{2}\,a_0 + \frac{k_1}{2k_0}\,(b_5 - a_0) + \frac{k_2}{k_0}\,(c_4 + a_0)\right)\,.
\end{aligned}
\tag{3.6}
$$

Obviously, these parameters depend only on a_0, a_2, b_3, b_4, b_5, c_3, c_4, and z_4. With (3.5) and (3.6) we can express the energy-momentum source of torsion and nonmetricity in the effective Einstein equation in terms of ϕ [27] (7.3, 7.5). This energy-momentum is exactly the energy-momentum of the Proca 1-form ϕ. Hence, we finally found that the MAG lagrangian (3.1) constrained by (3.2), together with its field equations, is effectively equivalent to an Einstein-Proca lagrangian as suggested in (3.4). The parameter m given in (3.6) has the meaning of the mass parameter of the Proca 1-form ϕ. If m vanishes, the initial MAG theory is equivalent to the Einstein-Maxwell theory. In general, $m = 0$ is equivalent to

$$
\begin{aligned}
0 = {} & 32\, b_4\, a_2\, b_3 - 12\, a_0\, a_2\, b_3 - 64\, b_4\, a_0\, b_3 - 24\, b_3\, {c_4}^2 - 48\, b_3\, c_4\, a_0 - 32\, b_4\, {a_0}^2 - 24\, b_4\, {c_3}^2 \\
& + 9\, a_2\, b_5\, a_0 - 6\, a_2\, {a_0}^2 + 9\, a_0\, {c_3}^2 - 48\, b_4\, c_3\, a_0 + 4\, b_4\, a_2\, a_0 - 24\, {a_0}^2\, c_4 + 9\, a_0\, {b_5}^2 \\
& - \frac{9}{2}\, a_2\, {b_5}^2 - 3\, a_0\, {c_4}^2 + 18\, c_3\, c_4\, b_5 - 18\, c_3\, c_4\, a_0 + 18\, c_3\, a_0\, b_5 + 18\, a_0\, c_4\, b_5\ . \qquad (3.7)
\end{aligned}
$$

This equation generalizes [28] (4.2) for $b_5 \neq 0$. Thus, the exact solution found in [28] with $m = 0$ corresponds to an exact solution of an Einstein-Maxwell system. Here, we want to present a solution for $m \neq 0$.

3.2 The Einstein-Proca system

Motivated by the previous section we now concentrate on the Proca lagrangian $\mathcal{L}_P$ of a massive 1-form ϕ:

$$
\mathcal{L}_P = -\frac{1}{2}\, d\phi \wedge {}^\star d\phi + \frac{1}{2}\, m^2 \phi \wedge {}^\star\phi\ . \qquad (3.8)
$$

First, we shortly discuss the dimension of m. We know that the Hodge-dual of a p-form is an $(n{-}p)$-form. Hence, whenever the Hodge-dual applies on a p-form, it has the dimension $[\,{}^\star\,] = \ell^{n-2p}$. We follow that $[d\phi \wedge {}^\star d\phi] = [\phi]^2$, and $\phi \wedge {}^\star\phi = \ell^2[\phi]^2$. To be able to consistently add these terms in the lagrangian the dimension of the mass parameter needs to be $[m] = 1/\ell$.

It is straightforward to calculate the Proca field equation and the canonical energy-momentum of this lagrangian with the Noether-Lagrange machinery presented in the appendix B or in [17]. The variations yield (following (B.9) and (B.19)):

$$
\frac{\delta \mathcal{L}_P}{\delta \phi} = d\frac{\partial \mathcal{L}_P}{\partial (d\phi)} + \frac{\partial \mathcal{L}_P}{\partial \phi} = -d^\star d\phi + m^2\, {}^\star\phi\ , \qquad (3.9)
$$

$$
\begin{aligned}
\frac{\delta \mathcal{L}_P}{\delta \vartheta^\alpha} =: \Sigma_\alpha &= e_\alpha \rfloor \mathcal{L}_P - (e_\alpha \rfloor d\phi)\frac{\partial \mathcal{L}_P}{\partial d\phi} - (e_\alpha \rfloor \phi)\frac{\partial \mathcal{L}_P}{\partial \phi} \\
&= \frac{1}{2}\left[-(e_\alpha \rfloor d\phi)\wedge {}^\star d\phi - d\phi \wedge (e_\alpha \rfloor {}^\star d\phi) + m^2 (e_\alpha \rfloor \phi)\wedge {}^\star\phi - m^2 \phi \wedge (e_\alpha \rfloor {}^\star\phi)\right] \\
&\quad + (e_\alpha \rfloor d\phi)\wedge {}^\star d\phi - m^2 (e_\alpha \rfloor \phi)\wedge {}^\star\phi \\
&= \frac{1}{2}\left[(e_\alpha \rfloor d\phi)\wedge {}^\star d\phi - d\phi \wedge (e_\alpha \rfloor {}^\star d\phi) - m^2\left[(e_\alpha \rfloor \phi)\wedge {}^\star\phi + \phi \wedge (e_\alpha \rfloor {}^\star\phi)\right]\right]\ .
\end{aligned}
\qquad (3.10)
$$

Coupled with a Riemannian background, i.e. considering a lagrangian $\mathcal{L} = \mathcal{L}_P + \mathcal{L}_{\text{Hilbert-Einstein}}$, we end up with the field equations

$$0 \stackrel{!}{=} -d^{\star}d\phi + m^2\,{}^{\star}\phi\,, \qquad \text{Proca equation}\,, \tag{3.11}$$

$$0 \stackrel{!}{=} G_\alpha - \kappa\Sigma_\alpha =: X_\alpha\,, \qquad \text{Einstein equation}\,. \tag{3.12}$$

For completeness we also display the contracted Bianchi identities

$$0 \stackrel{!}{=} d\Sigma^\alpha + \Gamma_\beta{}^\alpha \wedge \Sigma^\beta\,. \tag{3.13}$$

Also Obukhov and Vlachynsky [26] considered this system and indeed found the same solution we will find. Unfortunately, they did not publish their results until recently such that the author did not know about their efforts for a long time. However, the integration and the results are presented here in much more detail. Before we concentrate on a solution of this system, we consider two modified versions of the problem.

3.2.1 A Proca field on flat spacetime background

If we consider the lagrangian (3.8) on *flat*, i.e. Minkowskian background we find two simple solutions. The first we find by an ansatz in analogy to the static *electric* monopole field $A_{\text{el}} = q/r\,dt$: We suppose ϕ to be a static, spherically symmetric, and *time-like* 1-form

$$\phi = \frac{u_{\text{el}}(r)}{r}\,dt\,, \tag{3.14}$$

where t denotes the time coordinate and r the radius in spherical coordinates. With this ansatz, the Proca equation (3.11) becomes

$$0 \stackrel{!}{=} \frac{1}{r}\,(-u''_{\text{el}} + m^2\,u_{\text{el}}) \tag{3.15}$$

and can be solved by the well known Yukawa potential

$$\frac{u_{\text{el}}(r)}{r} = \frac{q}{r}\,\exp(-mr)\,. \tag{3.16}$$

The parameter q is called Proca charge. A second solution we find by considering an ansatz in analogy to the static *magnetic* monopole field $A_{\text{mag}} = p\,(1-\cos\theta)\,d\varphi$: We set

$$\phi = u_{\text{mag}}(r)\,(1-\cos\theta)\,d\varphi\,. \tag{3.17}$$

The field equations (3.11) becomes

$$0 \stackrel{!}{=} \frac{1}{r\,\sin\theta}\,(1-\cos\theta)(-u''_{\text{mag}} + m^2\,u_{\text{mag}}) \tag{3.18}$$

which is, of course, also solved by $u_{\text{mag}} = p\,\exp(-mr)$, where p might be called *magnetic* Proca charge.

3.2.2 A Proca vector field

Here we want to clarify that there *is* a difference between an ansatz of ϕ as a 1-form and as a vector. Consider the analogy of the lagrangian (3.8) for a vector-valued 0-form ϕ_α:

$$\mathcal{L}_P = -D\phi^\alpha \wedge {}^\star D\phi_\alpha \ + \ m^2 \phi^\alpha \wedge {}^\star \phi_\alpha \ , \tag{3.19}$$

$$\frac{\delta \mathcal{L}_P}{\delta \phi^\alpha} = -D\frac{\partial \mathcal{L}_P}{\partial (D\phi^\alpha)} + \frac{\partial \mathcal{L}_P}{\partial \phi^\alpha} = +2\, D^\star D\phi_\alpha + 2\, m^2\, {}^\star\phi_\alpha \ , \tag{3.20}$$

$$\begin{aligned}
\frac{\delta \mathcal{L}_P}{\delta \vartheta^\alpha} &=: \Sigma_\alpha = e_\alpha \rfloor \mathcal{L}_P - (e_\alpha \rfloor D\phi^\beta)\frac{\partial \mathcal{L}_P}{\partial D\phi^\beta} - (e_\alpha \rfloor \phi^\beta)\frac{\partial \mathcal{L}_P}{\partial \phi^\beta} \\
&= -(e_\alpha \rfloor D\phi^\beta) \wedge {}^\star D\phi_\beta + D\phi^\beta \wedge (e_\alpha \rfloor {}^\star D\phi_\beta) + m^2 (e_\alpha \rfloor \phi^\beta) \wedge {}^\star \phi_\beta + m^2 \phi^\beta \wedge (e_\alpha \rfloor {}^\star \phi_\beta) \\
&\quad + 2(e_\alpha \rfloor D\phi^\beta) \wedge {}^\star D\phi_\beta - 2m^2 (e_\alpha \rfloor \phi^\beta) \wedge {}^\star \phi_\beta \\
&= (e_\alpha \rfloor D\phi^\beta) \wedge {}^\star D\phi_\beta + D\phi^\beta \wedge (e_\alpha \rfloor {}^\star D\phi_\beta) - m^2 \left[(e_\alpha \rfloor \phi^\beta) \wedge {}^\star \phi_\beta - \phi^\beta \wedge (e_\alpha \rfloor {}^\star \phi_\beta)\right] \\
&= (e_\alpha \rfloor D\phi^\beta) \wedge {}^\star D\phi_\beta + D\phi^\beta \wedge (e_\alpha \rfloor {}^\star D\phi_\beta) + m^2 \phi^\beta \wedge (e_\alpha \rfloor {}^\star \phi_\beta) \ .
\end{aligned} \tag{3.21}$$

Note that we introduced the covariant derivative $D = d + \Gamma^{\{\}}$ with the Levi-Civita connection $\Gamma^{\{\}}$. In flat space, say, we find that the signs in (3.20) and (3.11) do not agree. Also, the energy-momentum (3.21) is very different from (3.10). From the introduction it is clear that we are interested in a Proca 1-form and not in a Proca vector field.

3.3 An solution with electric-type Proca field

Motivated by the Yukawa solution $\phi = \frac{q\ \exp(-mr)}{r}\,dt$ of the flat Proca equation and the analogy to the electric monopole, we make the ansatz that ϕ is a spherically symmetric, static, and time-like 1-form:

$$\phi = \frac{u(r)}{r} dt \ . \tag{3.22}$$

A first intuitive ansatz for the coframe (i.e. implicitly for the metric) would be of the Schwarzschild type

$$\begin{aligned}
&\vartheta^{\hat 0} = f\, dt \ , \quad \vartheta^{\hat 1} = \frac{1}{f}\, dr \ , \quad \vartheta^{\hat 2} = r\, d\theta \ , \quad \vartheta^{\hat 3} = r \sin\theta\, d\varphi \ , \quad f = f(r) \ , \\
&g \equiv \vartheta^{\hat 0} \otimes \vartheta^{\hat 0} - \vartheta^{\hat 1} \otimes \vartheta^{\hat 1} - \vartheta^{\hat 2} \otimes \vartheta^{\hat 2} - \vartheta^{\hat 3} \otimes \vartheta^{\hat 3} \ .
\end{aligned} \tag{3.23}$$

However, we find that the special component

$$Y := e^{\hat 1} \rfloor X^{\hat 0} \ + \ e^{\hat 0} \rfloor X^{\hat 1} \propto \frac{\kappa}{f^2 r^2} m^2 u^2 \stackrel{!}{=} 0 \tag{3.24}$$

of the Einstein equation restricts the class of solutions to the trivial case $\phi = 0$. Hence, we generalize the ansatz to the general spherically symmetric coframe

$$\vartheta^{\hat{0}} = f\, dt\,, \quad \vartheta^{\hat{1}} = \frac{g}{f}\, dr\,, \quad \vartheta^{\hat{2}} = r\, d\theta\,, \quad \vartheta^{\hat{3}} = r \sin\theta\, d\varphi\,, \quad f = f(r)\,, \quad g = g(r)\,. \tag{3.25}$$

The meaning of the function g is illustrated by the relation $\eta = g\, \eta_{\text{flat}}$ for the volume element $\eta = {}^\star 1$. Writing the Proca field as $\phi = \Phi(r)\, dt$, with $\Phi(r) := u(r)/r$, the energy-momentum (3.10) of this ansatz turns out to be

$$\begin{aligned}
\vartheta^{\hat{0}} \wedge \Sigma^{\hat{0}} &= \frac{\eta}{2}\left[-(\Phi'/g)^2 - m^2\Phi^2/f^2\right]\,,\\
\vartheta^{\hat{1}} \wedge \Sigma^{\hat{1}} &= \frac{\eta}{2}\left[\ \ (\Phi'/g)^2 - m^2\Phi^2/f^2\right]\,,\\
\vartheta^{\hat{2}} \wedge \Sigma^{\hat{2}} &= \frac{\eta}{2}\left[-(\Phi'/g)^2 - m^2\Phi^2/f^2\right]\,,\\
\vartheta^{\hat{3}} \wedge \Sigma^{\hat{3}} &= \frac{\eta}{2}\left[-(\Phi'/g)^2 - m^2\Phi^2/f^2\right]\,,\\
\vartheta^{\mu} \wedge \Sigma^{\nu} &= 0 \quad \text{for } \mu \neq \nu\,,\\
{}^\star(\vartheta^{\alpha} \wedge \Sigma_{\alpha}) &= -\frac{m^2\Phi^2}{f^2}\,,\\
{}^\star(\Sigma_{\alpha} \wedge {}^\star\Sigma^{\alpha}) &= \left(\frac{\Phi'}{g}\right)^4 + \left(\frac{\Phi' m\Phi}{fg}\right)^2 + \left(\frac{m\Phi}{f}\right)^4\,.
\end{aligned} \tag{3.26}$$

Recall that ${}^\star(\vartheta^\mu \wedge \Sigma^\nu) =: T^{\mu\nu}$ represent the components of the ordinary 2nd rank energy-momentum tensor. For Y defined in (3.24) we get

$$Y \propto \frac{1}{g^3 f^2 r^2}\left[g^3\kappa m^2 u^2 + 2\, g' f^4 r\right] \stackrel{!}{=} 0\,. \tag{3.27}$$

which now indeed has non-trivial solutions. The non-vanishing components of equations (3.11), (3.12), and (3.13) read in simplified form

$$\begin{aligned}
\mathrm{P}_0 &: \ 0 = f^2 g'(u - ru') + gr(f^2 u'' - g^2 m^2 u)\\
\mathrm{P}_1 &: \ 0 = g\dot{u} + \dot{g}(ru' - u)\\
\mathrm{E}_{00} &: \ 0 = -\kappa\left[(u - ru')^2 + m^2 r^2 u^2 g^2/f^2\right] + 2r^2(f^2 - g^2) + 4ff'r^3 - 4f^2 r^3 g'/g\\
\mathrm{E}_{11} &: \ 0 = -\kappa\left[(u - ru')^2 - m^2 r^2 u^2 g^2/f^2\right] + 2r^2(f^2 - g^2) + 4ff'r^3\\
\mathrm{E}_{01} &: \ 0 = f\dot{g} - g\dot{f}\\
\mathrm{E}_{22} &: \ 0 = -\kappa g f^4\left[(u - ru')^2 + m^2 r^2 u^2 g^2/f^2\right] + 2g^2 r^4 f(f\ddot{g} - g\ddot{f}) - 6\dot{f}(f\dot{g} - g\dot{f})\\
&\qquad + 2f^4 r^4 f'(fg' - gf') - 2f^5 f'' g r^4 + 2f^5 r^3(fg' - 2gf')\\
\mathrm{B}_0 &: \ 0 = f^3(ru' - u)\left[g\dot{u} + \dot{g}(ru' - u)\right] - g^2 m^2 r^2 u\left[g(f\dot{u} - u\dot{f}) + u(f\dot{g} - g\dot{f})\right]\\
\mathrm{B}_1 &: \ 0 = f^2 g'(u - ru') + gr(f^2 u'' - g^2 m^2 u)
\end{aligned} \tag{3.28}$$

Here, P, E, and B mark the Proca, Einstein, and contracted Bianchi equations and indices denote the respective components. These equations also include, for later purposes, the case of time dependent functions u, f, and g. Here, we neglect this time dependence and thus discard all time derivatives. We see that B_1 is equivalent to P_0 and that $\mathrm{E}_{11} - \mathrm{E}_{00} \propto Y$ simplifies the Einstein equations a lot. We end up with the following ordinary differential equations system of second order in u and first order in f and g:

$$\mathrm{P}_0: \quad 0 = f^2 g'(u - ru') + gr(f^2 u'' - g^2 m^2 u) \,, \tag{3.29}$$

$$\mathrm{E}_{11}: \quad 0 = -\kappa\left[(u - ru')^2 - m^2 r^2 u^2 g^2/f^2\right] + 2r^2(f^2 - g^2) + 4ff'r^3 \,, \tag{3.30}$$

$$\mathrm{E}_{11} - \mathrm{E}_{00}: \quad 0 = \kappa m^2 u^2 g^3 + 4f^4 r g' \,. \tag{3.31}$$

These equations are equivalent to [26] eqs (3.5, 3.9, 3.10) found by Obukhov and Vlachynsky. We did most of the calculations with the aid of the computer algebra systems Reduce and Maple, see appendix D.

3.3.1 Preparing the numerical integration

First we briefly discuss the dimensions of the system. We know the dimensions of the radius, the gravitational coupling constant, and the Proca mass, $[r] = \ell$, $[\kappa] = \ell^2$, $[m] = 1/\ell$, respectively. Hence, we get rid of all dimensions by rescaling the radius variable $r \to r/\sqrt{\kappa}$ and the mass parameter $m \to \sqrt{\kappa}\, m$. In practice, i.e. when investigating the equation system with computer algebra, we simply put $\kappa \equiv 1$, which is equivalent to the rescaling but saves us from introducing new variables. Also, we can eliminate the mass parameter m from (3.29-3.31) by the substitution $r \to m\, r$, $f \to f/m$, and $g \to g/m$. Instead, again, we equivalently fix $m = 1$ in the following without loosing generality.

We now concentrate on the dimensionless ordinary differential equation system (3.29-3.31). Parameter m is fixed. Thus it is *not* an integration constant. The equation system is of first order in f and g, and of second order in u. It is easy to reduce it to an ordinary first order differential equation system by substituting $u' \to v$, $u'' \to v'$ and by adding a fourth equation $v = u'$ to the system. Hence, a general integration of this system leads to four integration constants. In the case of a numerical approach, these constant are the initial values for f, g, u, and u' at the starting point of integration. Here, we consider only two starting points $r_0 \ll 1$ and $r_\infty \gg 1$ for integrations from zero and infinity. We denote the set of integration constants by (f_0, g_0, u_0, u_0') in one case and by $(f_\infty, g_\infty, u_\infty, u_\infty')$ in the other. The discussion above suggests that we will find a 4-parameter set of solutions. But this is misleading. In principle it is possible to start integration with four arbitrary parameters at some point $\tilde{r} \in \mathbb{R}$. But then one will find a solution of the equations (3.29-3.31) only in a neighborhood of $\tilde{r}$. In order to find global solutions for all $r \in \mathbb{R}$, we need to restrict the integration parameters: When we will soon investigate the limit $r \to 0$, we will only find a 3-parameter set of solutions. And later, when we will integrate this set of solutions to infinity, we will find that they diverge without a tuning of one parameter . Altogether, we are left with a 2-parameter set of global solutions.

First we discuss the constants for the limit r_∞. It is natural to require the metric function to tend to the Schwarzschild solution with the new mass parameter M that represents the total gravitating mass. Also, the Proca field should tend to the vacuum solution of the Proca equation, i.e. the Yukawa potential specified by the new Proca charge parameter Q. In detail we would expect

$$\begin{aligned} f_\infty &= \sqrt{1 - \frac{2M}{r_\infty}} \ , \\ g_\infty &= 1 \ , \\ u_\infty &= Q \ \exp(-mr_\infty) \ , \\ u'_\infty &= -mQ \ \exp(-mr_\infty) \ . \end{aligned} \tag{3.32}$$

Hence, we are left with *two* integration constants (M, Q).

For the limit $r \to 0$, we presume that g, u, and u' are finite at zero and that f diverges as $1/r$. This can be motivated by the fact that such an ansatz solves the system of equations (3.29-3.31) as we will show shortly, or by the arguments that g needs to be finite because of the volume element, u and u' should be finite because of the energy, and f should behave as $1/r$ because of the analogy to the Reissner-Nordström solution. Later, also our numerical integrations confirm this assumption. To investigate the limit $r \to 0$ we therefore insert the expansion

$$\begin{aligned} u &= u_1 + u_2\, r + u_3\, r^2 + .. \ , \\ g &= g_1 + g_2\, r + g_3\, r^2 + .. \ , \\ f &= f_1/r + f_2 + f_3\, r + .. \end{aligned} \tag{3.33}$$

into the system (3.29-3.31). Collecting the coefficients of 0-th order in r in each of the three equations we find respectively

$$\begin{aligned} 0 &= f_1\, g_2\, u_1 \\ 0 &= f_1^4\, g_1 - \frac{1}{2} h_1^2\, g_1\, u_1^2 \\ 0 &= h_1^4\, g_2 \ . \end{aligned} \tag{3.34}$$

These equations are solved by $g_2 = 0$ and $f_1 = u_1/\sqrt{2}$. This means that, for $r \to 0$, we find the following approximate solution of the equation system (3.29-3.31):

$$\begin{aligned} f_0 &= \frac{q}{\sqrt{2} r_0} \,, \\ g_0 &= c \,, \\ u_0 &= q \,, \\ u_0' &= b \,. \end{aligned} \tag{3.35}$$

Hence, for an integration from zero we are left *three* integration constants (q, b, c). Our results in section 3.3.4 will show that such solutions with arbitrary (q, b, c) diverge at some finite radius. To be more precise, the Proca function u diverges either to $+\infty$ or $-\infty$. Only a fine tuning of one of the parameters (we tuned c) makes the solution converging at infinity. Hence, regarding only converging, global solutions, we are also left with a 2-parameter class of solution parameterized by (q, b).

A power series expansion

There exists a simple scheme to determine all coefficients in the expansion (3.33): Inserting the expansion and considering the coefficients of i-th order ($i \geq 2$) in r in each of the three equations (3.29-3.31), one can solve for u_{i+2}, g_{i+2}, and f_{i+1} in terms of u_{j+2}, g_{j+2}, and f_{j+1} with $1 \leq j < i$. This iteration is very easily implemented in Maple and we display here the result after considering the equations up to 4-th order:

$$u(r) = q + b\, r + \frac{1}{10} \frac{c^2 \left(2\sqrt{2}\, f_2 - b\right)}{q^2} r^5 \,, \tag{3.36}$$

$$g(r) = c + \frac{1}{2} \frac{c^3}{q^2} r^4 - \frac{4}{5} \frac{c^3 \left(2\sqrt{2}\, f_2 - b\right)}{q^3} r^5 \,, \tag{3.37}$$

$$\begin{aligned} f(r) =& \frac{1}{\sqrt{2}} \frac{q}{r} + f_2 - \frac{1}{\sqrt{2}} \frac{f_2^2 - c^2}{q} r + \frac{\left(f_2^2 - c^2\right) f_2}{q^2} r^2 \\ & - \frac{1}{6\sqrt{2}} \frac{-2\, q^2\, c^2 + 15\, f_2^4 - 18\, f_2^2\, c^2 + 3\, c^4}{q^3} r^3 \,. \end{aligned} \tag{3.38}$$

Problems are that we cannot solve for f_2 in terms of q, b, and c, and that an insertion of the power expansion (3.33) into the equation system exceeds the computer's memory resources very fast. Being limited in this way, we could not observe an appropriate convergence behavior for large r.

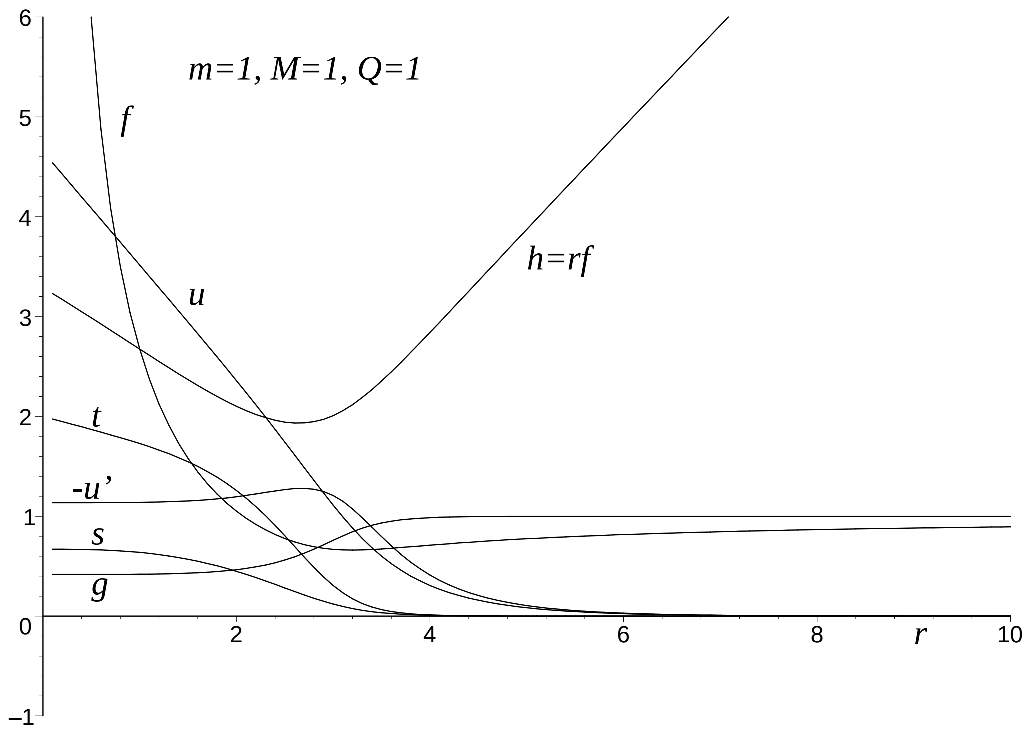


Table 3.1: Typical solution: Integration of (3.29-3.31) for $m = M = Q = 1$ from $r_\infty = 60$, with integration constants (3.32) performed by Maple. We see the metric functions f, g, and $h := r\,f$, the Proca function u and u', and the energy-momentum trace t and its integral s (divided by 100).

3.3.2 Integration from infinity for various M and Q

We perform the numerical integration with the standard Runge-Kutta method provided by the computer algebra system Maple. (In detail: We used the `rkf45` method with 15 digits, absolute (`abserr`) and relative (`relerr`) errors 10^{-13}, and unlimited number of function evaluations (`maxfun`).) You can find all calculations in the Maple-file given.

Table 3.1 represents a typical solution for an integration from infinity ($r_\infty = 60$, which is far enough from the Schwarzschild radius $r_S(M)$). The gravitating mass M and the Proca charge Q are of the same order as $m = 1$. We see the metric functions f and g. To get a better impression of the behavior of f for $r \to 0$ we add a plot of $h := rf$. The Proca function $u = \Phi/r$ exhibits a nicely localized density. Its derivative $-u'$ is less instructive. Also the energy-momentum trace of the Proca field

$$t := -{}^\star(\vartheta^\alpha \wedge \Sigma_\alpha) = m^2\Phi^2/f^2 = m^2u^2/h^2 \tag{3.39}$$

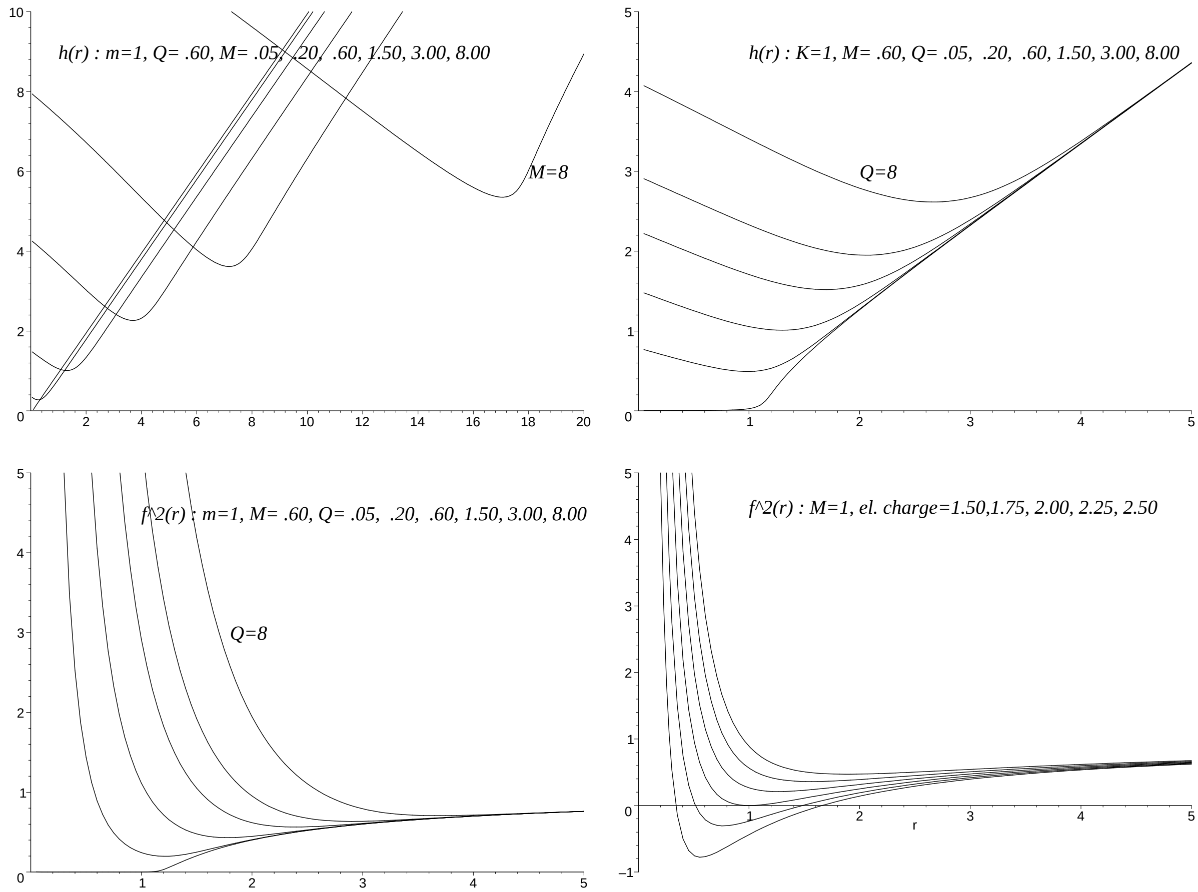


Table 3.2: Integration from r_∞. The metric functions $h = r\,f$ and f^2 are displayed for various M and Q. For a better comparison, we plot the Reissner-Nordström solution in the right bottom.

(see (3.26)) is quite localized. We also display its spatial integral

$$s(r) := 4\pi \int_r^\infty r^2 g(r) t(r)\, dr \quad \text{satisfying} \quad \int_{\text{spatial}} \vartheta^\alpha \wedge \Sigma_\alpha = -s(0)\, dt\ . \tag{3.40}$$

Next we vary M and Q. The upper two plots in table 3.2 show the metric function $h = rf$. The most interesting point of these plots is the following. As M is fixed and Q decreases, the metric function approaches the Schwarzschild behavior – but it never becomes imaginary! If f was the metric function of the Schwarzschild solution, then f^2 would become negative within the horizon and vanish at the horizon. Looking at the energy-momentum of the Proca field $t := m^2\Phi^2/f^2$, we can already follow that in our case f^2 may neither vanish nor be negative, as long as Φ is finite. Hence, a Proca solution with finite Φ *may not have a horizon.* For a better comparison, the lower two plots in table 3.2 display f^2 for our Proca

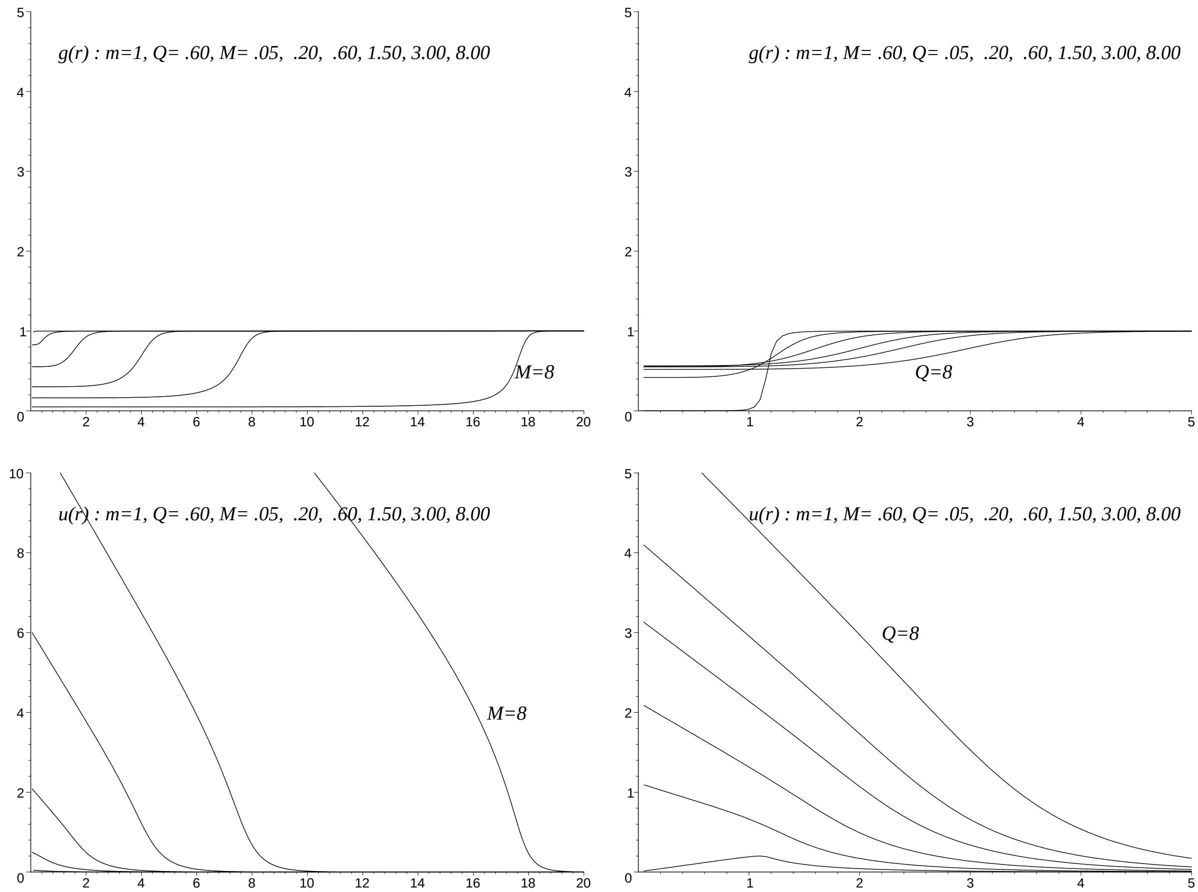


Table 3.3: Integration from r_∞. The metric function g and the Proca function u are displayed for various M and Q.

system (left plot) and the square metric function of the Reissner-Nordström solution (right plot) for varying Proca charge and electric charge, respectively. The Reissner-Nordström solution lacks a horizon as long as we choose the electric charge larger then $2M$, i.e. the *over extreme* case. For smaller charges the Reissner-Nordström solution has a horizon and the square metric function becomes negative. For our Proca system the behavior is similar for large Q. For smaller Q though, f^2 *approaches* zero but never becomes negative. The important result that our solution has no horizon is consistent with the analysis of Ayón-Beato et al. [9]. They proved that a static Einstein-Proca solution may not have a horizon by considering a spatial integral of the Proca equation. They called this a *no-hair* theorem for static black holes in the Einstein-Proca theory – or in the equivalent, constrained MAG theory.

In table 3.3 we plot the metric function g and the Proca function u. As we vary M, the metric function g seems to distinguish inside and outside regions. *Inside*, g takes some constant value within $[0,1]$ decreasing with increasing M. *Outside*, g equals 1. As we vary Q, we find that larger values for Q smear this boundary between inside and outside. Looking at the Proca function u, as we vary M, we find that the Proca field becomes perceptibly non-vanishing exactly within the same boundary g exhibits. Very interesting is the curve for $M = 0.6$ and $Q = 0.05$ in the right plots. The Proca field vanishes as $r \to 0$ and its derivative u' becomes positive. The metric function g approaches zero within the boundary instead of continuously approaching a finite $g(0)$ as it does for larger Q. This behavior is different indeed and belongs to *region II* as we will explain in the following section.

3.3.3 Comparing *internal* and *external* parameters

After the explicit presentation of the spherically symmetric solution of the Einstein-Proca system, we want to examine how the external parameters M and Q are correlated to the internal parameters q, b, and c. Both, Q and q, are in analogy to the Proca charge – but with respect to different limits $r \to \infty$ and $r \to 0$. How are they related? The computational power of Maple allows to integrate the system for a quite large array of values of M and Q. For this array we calculated the values of the internal parameters $q = u(0)$, $b = u'(0)$, and $c = g(0)$ and display them in table 3.4.

The first two of these plots display the internal Proca charge q. One can see that for any M, the internal q depends approximately linear on $\log_{10} Q$:

$$q = \alpha \log_{10} Q + \beta \,, \qquad \text{where roughly } 3 < \alpha < 4.5 \,. \tag{3.41}$$

In the white regions of the left plot, the numerical integration could not reach the requested accuracy of 10^{-13} (relative and absolute error). The next two plots in table 3.4 show b and c. The noisy peaks in the plot of b are at the very edge to regions where the integration could not reach the requested accuracy. However, we observe a smooth transition to positive values of b. This *region II* belongs to small values of M and Q as the diagram at the bottom illustrates. Also the plot of c clearly demonstrates this edge to region II but at the same time exhibits the smooth transition to this region for smaller Q. We cannot completely exclude that this behavior is an artifact of the numerical integration in the case of a too small deviation from the Schwarzschild configuration for large r.

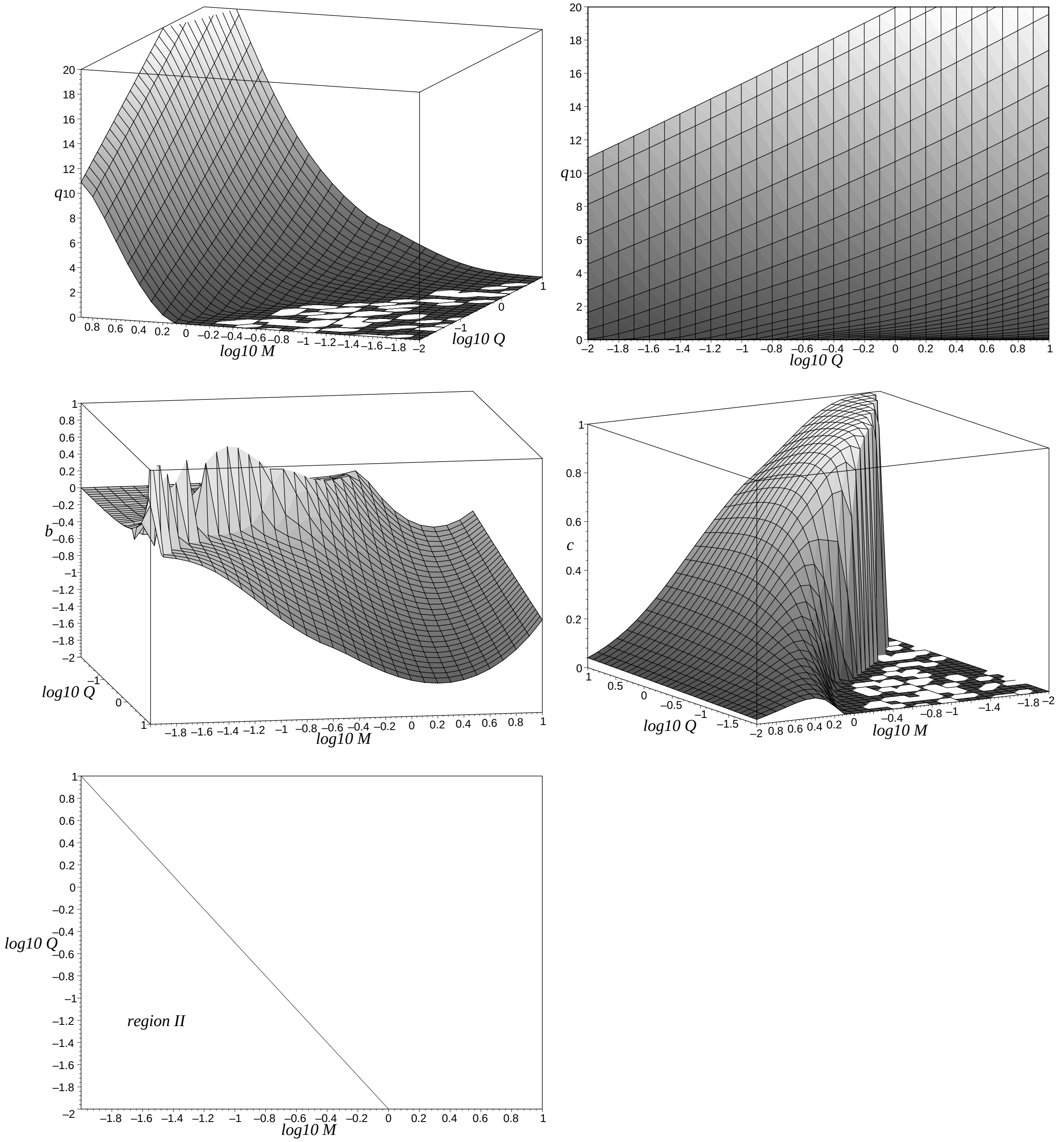


Table 3.4: The field configuration $q = u(0)$, $b = u'(0)$, and $c = g(0)$ at zero is displayed for an array $[-2 < \log_{10} M < 1, -2 < \log_{10} Q < 1]$ of different values for the external parameters M and Q.

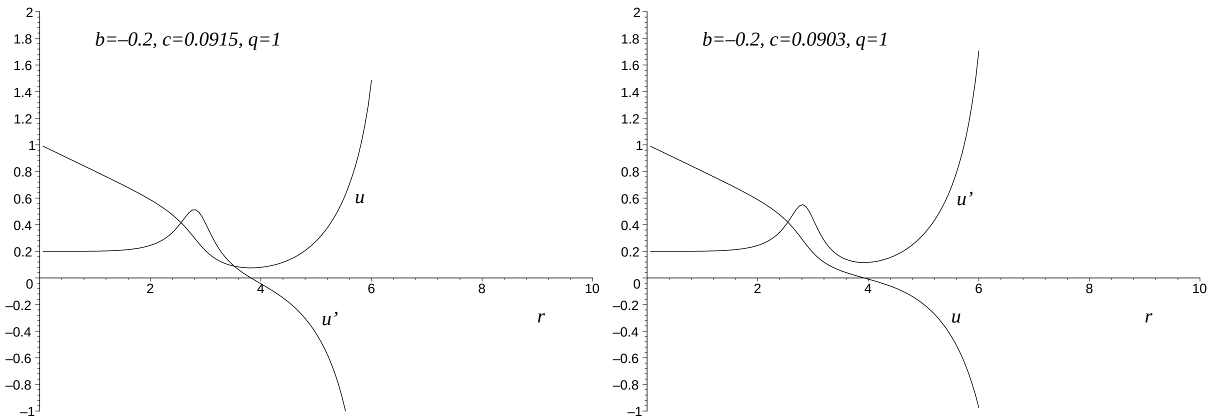


Table 3.5: The integration from zero for arbitrary constants q, b, and c: If c is too large (left plot) the Proca function u diverges to $+\infty$, whereas if c is too small (right plot) u diverges to $-\infty$. Note the small difference of 0.0012 between the two values of c.

3.3.4 Solutions with fixed internal Proca charge

As we already mentioned, the integration from zero is quite costly. If we start integration with the constants (3.35) with arbitrary q, b, and c, the solution diverges at some finite radius. Table 3.5 displays the two possible divergences: the Proca function u either diverges to $+\infty$, if c is large, or to $-\infty$, if c is small. Only a fine tuning of c allows to find a global solution by an integration from zero. Of course, these integrations exhibit the same solution as an integration from ∞. Since this procedure is very time expensive, we had to find another way to produce solutions with fixed internal Proca charge by using the relation (3.41) between q and Q. We fixed q by tuning Q for given M. This may be done very quickly because if we have found one solution with arbitrary q, relation (3.41) tells us of how to approximately choose Q for a given value of q. Thereby we need only about five steps to fix q on a given value up to an accuracy of 10^{-5}. Table 3.6 shows how Q has to be chosen for different M in order to fix $q = 1$. It almost seems to be a linear relation between $\log_{10} Q$ and $\log_{10} M$. Table 3.7 displays the solutions for fixed $q = 1$ and different M. The plot of h nicely demonstrates the necessary relation $h(0) = q/\sqrt{2}$ we found in (3.35). The plot of f^2, again, demonstrates that our solutions *do not have a horizon.* Instead, f^2 approaches zero but never becomes negative. Finally, the plots of g and u exhibit the localization of our Proca particle within a finite radius.

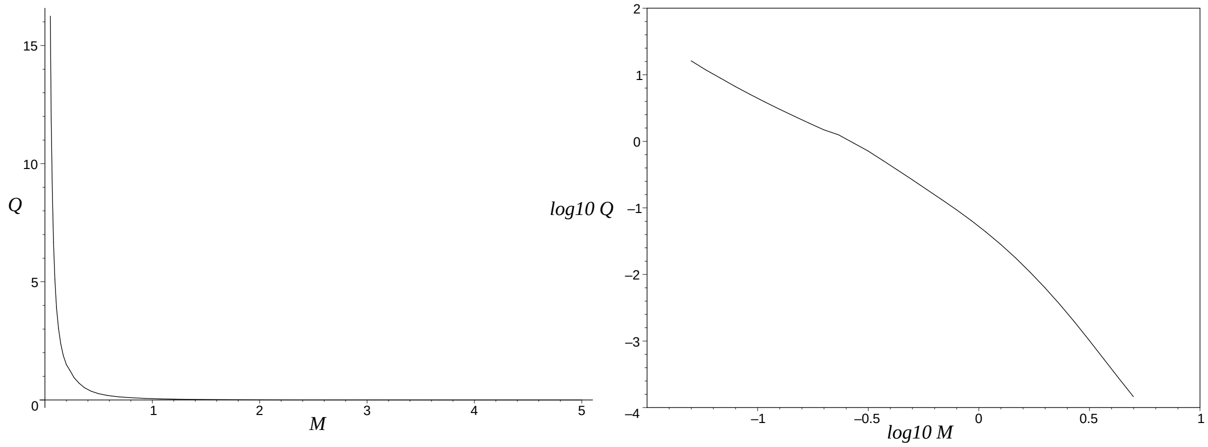


Table 3.6: How to choose Q for given M in order to fix $q = 1$.

3.3.5 Testing the stability

The fact that our solution has a naked singularity at the origin suggests that it may not be stable. To investigate its stability we follow the scheme Jetzer [19] developed for the case of boson stars. In this approach we insert small, time-dependent, and spherically symmetric perturbations in all function f, g, and u:

$$\begin{aligned} u(r) &\to u(r) + \epsilon\, \delta u(r,t)\ , \\ g(r) &\to g(r) + \epsilon\, \delta g(r,t)\ , \\ f(r) &\to f(r) + \epsilon\, \delta f(r,t)\ . \end{aligned} \tag{3.42}$$

We now consider the Einstein-Proca equations (3.28) that include the case of time dependent functions u, f, and g. Inserting the perturbation (3.42) and discarding terms of $O(\epsilon^2)$ we find

$$\mathrm{P}_1 :\ 0 = g\ \delta\dot{u} + \delta\dot{g}\ (ru' - u)\ , \tag{3.43}$$

$$\mathrm{E}_{01} :\ 0 = f\ \delta\dot{g} - g\ \delta\dot{f}\ , \tag{3.44}$$

$$\mathrm{B}_0 :\ 0 = f\ \delta\dot{u} - u\ \delta\dot{f}\ . \tag{3.45}$$

These equations may easily be integrated:

$$\mathrm{P}_1 \Rightarrow\ \delta g = C_1(r) + \frac{g}{u - ru'}\,\delta u\ , \tag{3.46}$$

$$\mathrm{E}_{01} \Rightarrow\ \delta f = C_2(r) + \frac{f}{g}\,\delta g\ , \tag{3.47}$$

$$\mathrm{B}_0 \Rightarrow\ \delta f = C_3(r) + \frac{f}{u}\,\delta u\ . \tag{3.48}$$

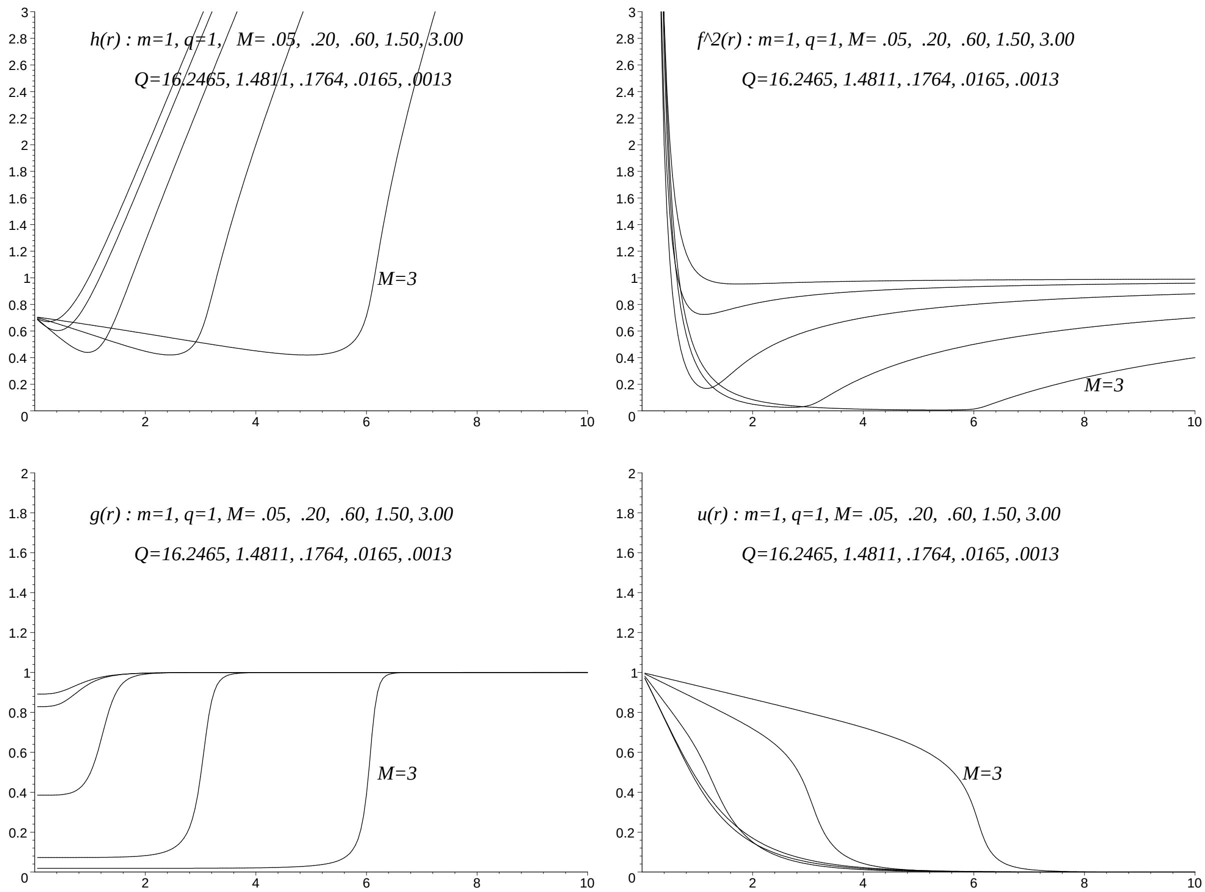


Table 3.7: The metric functions $h = r\,f$, f^2, and g and the Proca function u for fixed internal Proca charge $q = 1$.

This, however, gives two expressions for δf:

$$
\begin{aligned}
& C_3(r) + \frac{f}{u}\;\delta u = \delta f = C_2(r) + \frac{f}{g}\;\left(C_1(r) + \frac{g}{u - r u'}\;\delta u\right) \\
\Leftrightarrow \qquad & \delta u \left(\frac{f}{u} - \frac{f}{u - r u'}\right) = C_2(r) + \frac{f}{g}\;C_1(r) - C_3(r) \qquad (3.49)
\end{aligned}
$$

We need to conclude that δu may be expressed as a function of r only, i.e. it may not be time dependent. This sabotages the sense of our ansatz and prohibits to draw any conclusions from this analysis. Following Jetzer, we should assume a time dependence $\delta u = \exp(i\sigma t) v(r)$ and find an eigenequation $F\;v = \sigma^2\;v$, where F is a differential operator of second order in r depending on the function u, f, and g and their derivatives. A negative lowest eigenvalue of σ^2 would mean instability.

3.4 Other approaches

Failure of the magnetic-type ansatz

In section 3.2.1 we found a solution $\phi = p \exp(-mr)\,(1 - \cos\theta)\,d\varphi$ for the flat Proca equation (3.11). However, with the general spherically symmetric ansatz (3.25) for the coframe, we find the 12-component of the Einstein equation:

$$\vartheta^1 \wedge X^2 = \frac{f\kappa u u'(\cos\theta - 1)}{g r^3 \sin\theta}\,\eta \tag{3.50}$$

which has only trivial solutions. Hence, there exists no magnetic analogue to the previous solution!

Rosen's ansatz

In this section we briefly discuss the ansatz of Rosen [31]. He considered the lagrangian (3.8) of a Proca 1-form (although talking about a Proca vector field) together with the ansatz

$$\phi = w_0(r)\mathrm{e}^{-i\omega t}\,dt + w_1(r)\mathrm{e}^{-i\omega t}\,dr\ . \tag{3.51}$$

With (3.51) and the spherically symmetric coframe (3.25), the Proca equation (3.11) reads

$$\begin{aligned} 0 = \vartheta^0 \wedge \vartheta^2 \wedge \vartheta^3\ &\frac{\mathrm{e}^{-i\omega t}}{fg}\,\left[w_1(f^2m^2 - \omega^2) - i\omega w_0'\right] \\ - \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3\ &\frac{f\mathrm{e}^{-i\omega t}}{g^2 r}\,\Big[2i\omega w_1 + g^2m^2 r w_o/f^2 - 2w_0' \\ &\qquad - (w_0'' - i\omega w_1')r + (w_0' - i\omega w_1)rg'/g\Big]\ . \end{aligned} \tag{3.52}$$

From the 023-component we read off

$$w_1 = \frac{i\omega w_0'}{f^2m^2 - \omega^2}\ , \tag{3.53}$$

which is equivalent to equation [31] (17). We substitute w_1 and identify $w \equiv w_0$. *In flat space*, the 123-component of the Proca equation (3.52) becomes

$$0 = rw'' + 2w' + (\omega^2 - m^2)rw\ , \tag{3.54}$$

which is in agreement with [31] (19). Comparing with (3.15), we find that this equation is the same as the Proca equation in flat space for a Proca 1-form $w(r)\,dt$ with mass parameter $\sqrt{m^2 - \omega^2}$. Hence, if we set $\omega = m$, as Rosen proposes in equation [31] (33) (in his notation $C = 1 \Rightarrow \omega = \kappa$), then (3.54) is the ordinary, *massless* Maxwell equation. Thus, in flat spacetime, such a particle has no finite extension. This raises the question of how to choose the initial constraints at infinity for such a numerical integration of the field equations. Finally, Rosen assumes that the total gravitating mass (M in our notation) is equal to the mass parameter m (in dimensionless units). This is similar to equation (2.8) we found when discussing *dimensions* but not *quantities*. In general, one can hardly compare Rosen's work with our's because he concentrates on the idea of an *elementary particle with finite and absolute boundary* existing in the Einstein-Proca theory. Thus, he assumes an *empty* (exactly Schwarzschild) space outside the particle's boundary – and not a space that becomes Schwarzschild asymptotically, as we did. He calculates the solution by continuously (not smoothly) fitting the (Einstein-Proca) fields inside to the (purely Einstein) fields outside.

3.5 Summary

The introduction of this chapter explained the meaning of the coupled Einstein-Proca theory as an effective theory of MAG and thus motivated our analysis of this theory. We noticed that the discussion of the breaking of the linear group also insinuates this equivalence. Most interesting, we found the general condition (3.7) for the massless case, i.e. for the (constrained) lagrangian being equivalent to the Einstein-Maxwell theory. Then we revisited the variational procedure in appendix B, derived the field equations (3.11, 3.12) and the energy-momentum (3.10) of the Einstein-Proca theory, and displayed the (electric- and magnetic-type) Yukawa solution in *flat* spacetime. For an electric-type ansatz we discussed the numerical integration and its integration constants and also offered the power series expansion (3.38) at the origin. We also proved the failure of the magnetic-type ansatz. Here, we collect the essential features of the numeric solution:

(1) In table 3.1 we display the typical solution of the Einstein-Proca system for the case of the gravitating mass M and the external Proca charge Q being of the same order as the mass parameter m.

(2) Table 3.2 concentrates on the behavior of the metric function f, with $\vartheta^{\hat{0}} = f\,dt$. We found that our solution has *no horizon*, which should also be clear from the energy-momentum trace in (3.26) and is consistent with [9]. Hence, our solution has a naked singularity.

(3) Table 3.3 focuses on the shape of the Proca particle. We found some boundary which

is sharp for small external Proca charge Q. The larger the gravitating mass M, the larger the extension of the Proca particle.

(4) Table 3.4 exhibits the interesting linear relation (3.41) between the internal Proca charge q and the logarithm of the external Proca charge $\log_{10} Q$.

(5) The b- and c-plots in table 3.4 and the g- and u-plots in table 3.3 suggest a different kind of behavior for small M and Q (region II). Although the transition to this behavior is smooth, it might still be an artifact of the numerical integration.

Finally, we want to add a remark that relates to the previous chapter in the context of MAG. We saw that a special case of MAG is equivalent to the Einstein-Proca theory with Proca mass parameter m given in (3.6). We are now interested in monopole charges of the *strong gravity field*, which is, in our case, only the dilation curvature $R_\gamma{}^\gamma$. This curvature part can be identified with the *Proca field strength* via $d\phi = \frac{1}{k_0} dQ_\gamma{}^\gamma = \frac{1}{k_0} R_\gamma{}^\gamma$ (cf. (3.5)) and hence we should investigate $d\phi$ for monopoles. In the massive case $m \neq 0$, the Proca field is of short range (i.e. $u(r) \stackrel{r\to\infty}{\longrightarrow} \exp(-mr)$) and does not carry monopole charges. In the massless case $m = 0$ (cf. (3.7)) though, ϕ represents an electromagnetic potential and the field strength $d\phi$ may carry electric monopole charges as well as magnetic. Hence, it is simple to construct MAG solutions with monopole charges in the dilation field by translating back the corresponding Reissner-Nordström solutions.

Appendix A

Conventions and identities

Table A.1 displays all operators we use in the context of the exterior algebra and Lie groups. Table A.2 clarifies our conventions concerning the indices of vectors, matrices, and tensors.

A.1 The exterior calculus and the hodge dual

[*A.1*] Given a coframe ϑ^α, i.e. a basis of a dual vector space V^*, we define

$$\vartheta^{\alpha..\beta} := \vartheta^\alpha \wedge .. \wedge \vartheta^\beta \ . \tag{A.1}$$

[*A.2*] If a metric g and a coframe ϑ^α is given, we define the components $g_{\alpha\beta}$, the components $g^{\alpha\beta}$, and the determinant $|g|$ of the metric as

$$g = g_{\alpha\beta}\, \vartheta^\alpha \otimes \vartheta^\beta \ , \tag{A.2}$$

$$g^{\alpha\beta}\, g_{\beta\gamma} = \delta^\alpha_\gamma \ , \tag{A.3}$$

$$|g| = \det\ g_{\alpha\beta} \ . \tag{A.4}$$

Indices are raised and lowered by the metric components via $\psi^\alpha = g^{\alpha\beta}\ \psi_\beta$ and $\psi_\alpha = g_{\alpha\beta}\ \psi^\beta$.

[*A.3*] We define the generalized δ-symbol by

$$\delta^{cd..}_{ab..} := \begin{cases} \mathrm{sgn}(\pi) & \text{if there exists a permutation } \pi \text{ with } \pi(ab..) = (cd..) \\ 0 & \text{else} \end{cases} \ . \tag{A.5}$$

d	The exterior derivative.
$\wedge$	The exterior product.
$\rfloor$	The inner product.
$\pounds$	The Lie derivative $\pounds_\xi \psi = \xi\rfloor(d\psi) + d(\xi\rfloor\psi)$.
$\star$	The hodge dual defined by (A.9).
$\cdot$	The action of some group or group generators on a representation space.
$\circ$	The multiplication of two group elements or the concatenation of two generators.
$\overset{\circ}{\wedge}$	The exterior product and concatenation of two algebra-valued forms.
$\dot{\wedge}$	The exterior product and action of an algebra-valued form on a representation space-valued form.
$[\ ,\ ]$	The multiplication $[X,Y] = X\circ Y - Y\circ X$ in a Lie algebra.
$\langle\ ,\ \rangle$	A metric in a vector space, usually Lie algebra.

Table A.1: Conventions of operators.

We define the components $\epsilon_{a_1..a_n}$ and $\epsilon^{a_1..a_n}$ of the ϵ-tensor as

$$\epsilon_{a_1..a_n} := \delta^{1..n}_{a_1..a_n} \ , \tag{A.6}$$

$$\epsilon^{b_1..b_n} := \delta^{b_1..b_n}_{1..n} \ . \tag{A.7}$$

Hence, the indices of the ϵ-tensor are *not* raised by a metric tensor! It follows

$$\epsilon_{12..n} = +1 \ , \quad \epsilon^{12..n} = +1 \ , \quad \epsilon_{a_1..a_n}\,\epsilon^{b_1..b_n} := \delta^{b_1..b_n}_{a_1..a_n} \ . \tag{A.8}$$

[*A.4*] Given a metric g we define the Hodge dual of a p-form ψ in n-dimensional space ($p \leq n$) as (equivalent to [4] eq (7.173))

$$^\star\psi := \frac{\sqrt{|g|}}{(n-p)!\,p!}\ \psi^{\alpha_1..\alpha_p}\ \epsilon_{\alpha_1..\alpha_n}\ \vartheta^{\alpha_{p+1}..\alpha_n} \tag{A.9}$$

$$\Leftrightarrow \quad (^\star\psi)_{\alpha_{p+1}..\alpha_n} := \frac{\sqrt{|g|}}{p!}\ \psi^{\alpha_1..\alpha_p}\ \epsilon_{\alpha_1..\alpha_n} \ , \tag{A.10}$$

$$\Leftrightarrow \quad ^\star(\vartheta_{\alpha_1..\alpha_p}) := \frac{\sqrt{|g|}}{(n-p)!}\ \epsilon_{\alpha_1..\alpha_n}\ \vartheta^{\alpha_{p+1}..\alpha_n} \ . \tag{A.11}$$

$()^a, ()^i, ()^\alpha$	The components of a vector x with respect to a basis e_a of a vector space V such that $x = x^a e_a \in V$. ($()^\alpha$ represents a *column* vector.) If V is a spacetime tangent space we usually introduce a natural basis $\partial_i \in T_pM$ or an anholonomic basis $e_\alpha(p) \in T_pM$. For a Lie algebra we introduce the basis $\lambda_a \in \mathcal{G} \simeq T_{\mathrm{id}}G$ of generators.
$()_a$	The components of a 1-form ψ in the dual vector space V^* with respect to a cobasis ϑ^a such that $\psi = \psi_a\vartheta^a \in V^*$. ($()_\alpha$ represents a *row* vector.)
$()^{a_1..a_s}{}_{b_1..b_t}$	The components of a (s,t)-tensor T (of *contravariant* rank s and *covariant* rank t) with respect to a basis e_a and a cobasis ϑ^b such that $T = T^{a_1..a_s}{}_{b_1..b_t}\, e_{a_1} \otimes .. \otimes e_{a_s} \otimes \vartheta^{b_1} \otimes .. \otimes \vartheta^{b_t} \in (\bigotimes_s V) \otimes (\bigotimes_t V^*)$.
$()_b{}^a$	The matrix-components of a linear map $A:\ x^a \to A_b{}^a\, x^b$ where b and a are the column and row indices, respectively.
$()^a{}_b$	The components of the inverse map $A^{-1}:\ x^a \to A^a{}_b\, x^b$ with $A^a{}_b = (A^{-1})_b{}^a$ and $A^a{}_b A_c{}^b = \delta^a_c$. It follows $(B^{-1}AB)_b{}^a = B^a{}_d A_c{}^d B_b{}^c$.
$\mathring{A}$, $\hat{A}$, $\not{A}$	The symmetric, antisymmetric, and tracefree-symmetric part of a matrix only defined via a given metric g_{ab} that enables to raise and lower indices: $\mathring{A}_{ab} := A_{(ab)} := \frac{1}{2}(A_{ab} + A_{ba})$, $\hat{A}_{ab} := A_{[ab]} := \frac{1}{2}(A_{ab} - A_{ba})$, $\not{A}_{ab} := A_{(ab)} - 1/n\, g_{ab}\, A_c{}^c$.

Table A.2: Conventions of indices and symmetries.

Note that in A.9, in order to raise the indices of ψ, we make use of the metric tensor. We define the η-basis as

$$\eta^{\alpha_1..\alpha_p} := {}^\star(\vartheta^{\alpha_1..\alpha_p})\ . \tag{A.12}$$

The dimension of the hodge operator * in n dimensions when applied on a p-form is $[{}^\star] = \ell^{n-2p}$. If ψ is a p-form and 'ind' denotes the number of minus signs in the diagonal form of the metric, we find the identities (cf. [24]):

$$ {}^{\star\star}\psi = (-)^{p(n-p)+\text{ind}}\ \psi\ , \tag{A.13} $$

$$ {}^{\star}\psi \wedge \phi = {}^{\star}\phi \wedge \psi\ , \tag{A.14} $$

$$ e_\alpha \rfloor {}^{\star}\psi = {}^{\star}(\psi \wedge \vartheta_\alpha)\ , \tag{A.15} $$

$$ {}^{\star}(e_\alpha \rfloor \psi) = (-)^{p-1}\ \vartheta_\alpha \wedge {}^{\star}\psi\ , \tag{A.16} $$

$$ \vartheta^\mu \wedge (e_\mu \rfloor \psi) = p\,\psi\ , \tag{A.17} $$

$$ e_\mu \rfloor \eta^{\alpha_1..\alpha_p} = \eta^{\alpha_1..\alpha_p}{}_\mu\ , \tag{A.18} $$

$$ \vartheta^\mu \wedge \eta^{\alpha_1..\alpha_p} = \sum_{i=1}^{p} (-)^{p-i} g^{\mu\alpha_i}\ \eta^{\alpha_1..\alpha_{i-1}\alpha_{i+1}..\alpha_p}\ , \tag{A.19} $$

and consequently

$$ \begin{aligned} e_\alpha \rfloor \psi &\overset{(A.13)}{=} (-)^{p(n-p)+\text{ind}}\ e_\alpha \rfloor {}^{\star\star}\psi \\ &\overset{(A.15)}{=} (-)^{p(n-p)+\text{ind}}\ {}^{\star}({}^{\star}\psi \wedge \vartheta_\alpha) \\ &= (-)^{p(n-p)+(n-p)+\text{ind}}\ {}^{\star}(\vartheta_\alpha \wedge {}^{\star}\psi) \\ &= (-)^{n(p+1)+\text{ind}}\ {}^{\star}(\vartheta_\alpha \wedge {}^{\star}\psi)\ , \end{aligned} \tag{A.20} $$

$$ \begin{aligned} {}^{\star}(e_\alpha \rfloor {}^{\star}\psi) &\overset{(A.15)}{=} {}^{\star\star}(\psi \wedge \vartheta_\alpha) \\ &\overset{(A.13)}{=} (-)^{(p+1)(n-(p+1))+\text{ind}}\ (\psi \wedge \vartheta_\alpha) \\ &= (-)^{(n+1)(p+1)+\text{ind}}\ (\psi \wedge \vartheta_\alpha)\ . \end{aligned} \tag{A.21} $$

Note that eqs (A.20) and (A.21) correct eqs (98b) and (98d) in [24] which are valid for *even* n only.

[$A.5$] If we consider the somewhat strange definition of the inner product of two forms as given in [18]:

$$ \phi \rfloor \psi := {}^{\star}(\phi \wedge {}^{\star}\psi)\ , \tag{A.22} $$

we find

$$ \begin{aligned} \vartheta_\alpha \rfloor \vartheta^\mu &\overset{(A.22)}{=} {}^{\star}(\vartheta_\alpha \wedge {}^{\star}\vartheta^\mu) \\ &= (-)^{n-1}\ {}^{\star}({}^{\star}\vartheta^\mu \wedge \vartheta_\alpha) \\ &\overset{(A.15)}{=} (-)^{n-1}\ (e_\alpha \rfloor {}^{\star\star}\vartheta^\mu) \\ &\overset{(A.13)}{=} (-)^{n-1}(-)^{1(n-1)+\text{ind}}\ (e_\alpha \rfloor \vartheta^\mu) \\ &= (-)^{\text{ind}}\ \delta^\mu_\alpha\ , \end{aligned} \tag{A.23} $$

$$\begin{aligned}
{}^\star\Big[\delta\vartheta_\mu\rfloor(\vartheta^\mu\wedge\psi)\Big] &\overset{(A.22)}{=} {}^{\star\star}\Big[\delta\vartheta_\mu\wedge{}^\star(\vartheta^\mu\wedge\psi)\Big] \\
&\overset{(A.13)}{=} (-)^{(n-p)(n-(n-p))+\text{ind}}\;\delta\vartheta_\mu\wedge{}^\star(\vartheta^\mu\wedge\psi) \\
&= (-)^{(n-p)p+\text{ind}}(-)^p\;\delta\vartheta_\mu\wedge{}^\star(\psi\wedge\vartheta^\mu) \\
&\overset{(A.15)}{=} (-)^{(n-p+1)p+\text{ind}}\;\delta\vartheta_\mu\wedge(e^\mu\rfloor{}^\star\psi)\;. \qquad \text{(A.24)}
\end{aligned}$$

[*A.6*] We now cite some variation rules for the Hodge dual out of [24]. We are given a metric g, a coframe ϑ^α, two arbitrary forms ψ_1 and ψ_2, and a p-form ψ. For the variation we generally require

$$\delta(\psi_1\wedge\psi_2) = \delta\psi_1\wedge\psi_2 + \psi_1\wedge\delta\psi_2\;, \qquad \text{(A.25)}$$

$$[d,\delta] = 0\;. \qquad \text{(A.26)}$$

It follows

$$\delta\vartheta^\mu\wedge(e_\mu\rfloor\psi) = \frac{1}{p!}\,\psi_{\alpha_1..\alpha_p}\,\delta\,\vartheta^{\alpha_1..\alpha_p}\;, \qquad \text{(A.27)}$$

$$\delta g^{\alpha\beta} = -g^{\alpha\gamma}\,g^{\beta\delta}\,\delta g_{\gamma\delta}\;, \qquad \text{(A.28)}$$

$$\delta|g| = |g|\,g^{\alpha\beta}\,\delta g_{\alpha\beta}\;, \qquad \text{(A.29)}$$

$$\delta\eta^{\alpha_1..\alpha_p} = \delta\vartheta^\mu\wedge(e_\mu\rfloor\eta^{\alpha_1..\alpha_p}) + \delta g_{\beta\gamma}\left[\vartheta^\beta\wedge\eta^{\alpha_1..\alpha_p\gamma} - \frac{1}{2}\,g^{\beta\gamma}\,\eta^{\alpha_1..\alpha_p}\right], \qquad \text{(A.30)}$$

$$(\delta^\star - {}^\star\delta)\psi = \delta\vartheta^\alpha\wedge(e_\alpha\rfloor{}^\star\psi) - {}^\star\big[\delta\vartheta^\alpha\wedge(e_\alpha\rfloor\psi)\big] + \delta g_{\alpha\beta}\Big[\vartheta^{(\alpha}\wedge(e^{\beta)}\rfloor{}^\star\psi) - \frac{1}{2}\,g^{\alpha\beta}\,{}^\star\psi\Big]\;. \qquad \text{(A.31)}$$

Hence, the condition $\delta^\star\psi = {}^\star\delta\psi$ for an arbitrary p-form ψ is equivalent to the relation

$$\delta g_{\alpha\beta} = -2g_{\gamma(\alpha}\,e_{\beta)}\rfloor\delta\vartheta^\gamma = -2\omega_{(\alpha\beta)}\,, \qquad \text{where} \qquad \delta\vartheta^\gamma = \omega_\delta{}^\gamma\vartheta^\delta\;. \qquad \text{(A.32)}$$

Therefore, for an orthonormal coframe, the allowed variations are of the Lorentz type, i.e., $\omega_{(\alpha\beta)} \equiv 0$.

A.2 Some calculations

[*A.7*] Let $\Gamma_\alpha{}^\beta$ be a matrix-valued 1-form and let a metric be given that allows to raise and lower indices. We investigate some symmetries of composed terms for section 1.4.4.

$$
\begin{aligned}
&\hat\Gamma_{\gamma\beta}\wedge\hat\Gamma_\alpha{}^\gamma = -\hat\Gamma_\alpha{}^\gamma\wedge\hat\Gamma_{\gamma\beta} = \quad \hat\Gamma_\alpha{}^\gamma\wedge\hat\Gamma_{\beta\gamma} = -\hat\Gamma_{\alpha\gamma}\wedge\hat\Gamma_\beta{}^\gamma = -\hat\Gamma_{\gamma\alpha}\wedge\hat\Gamma_\beta{}^\gamma = \hat\Gamma_{\gamma[\beta}\wedge\hat\Gamma_{\alpha]}{}^\gamma \\
&\mathring\Gamma_{\gamma\beta}\wedge\mathring\Gamma_\alpha{}^\gamma = -\mathring\Gamma_\alpha{}^\gamma\wedge\mathring\Gamma_{\gamma\beta} = -\mathring\Gamma_\alpha{}^\gamma\wedge\mathring\Gamma_{\beta\gamma} = -\mathring\Gamma_{\alpha\gamma}\wedge\mathring\Gamma_\beta{}^\gamma = -\mathring\Gamma_{\gamma\alpha}\wedge\mathring\Gamma_\beta{}^\gamma = \mathring\Gamma_{\gamma[\beta}\wedge\mathring\Gamma_{\alpha]}{}^\gamma \\
&\hat\Gamma_{\gamma\beta}\wedge\mathring\Gamma_\alpha{}^\gamma = -\mathring\Gamma_\alpha{}^\gamma\wedge\hat\Gamma_{\gamma\beta} = \quad \mathring\Gamma_\alpha{}^\gamma\wedge\hat\Gamma_{\beta\gamma} = \quad \mathring\Gamma_{\alpha\gamma}\wedge\hat\Gamma_\beta{}^\gamma = \quad \mathring\Gamma_{\gamma\alpha}\wedge\hat\Gamma_\beta{}^\gamma \\
&\mathring\Gamma_{\gamma\beta}\wedge\hat\Gamma_\alpha{}^\gamma = -\hat\Gamma_\alpha{}^\gamma\wedge\mathring\Gamma_{\gamma\beta} = -\hat\Gamma_\alpha{}^\gamma\wedge\mathring\Gamma_{\beta\gamma} = -\hat\Gamma_{\alpha\gamma}\wedge\mathring\Gamma_\beta{}^\gamma = \quad \hat\Gamma_{\gamma\alpha}\wedge\mathring\Gamma_\beta{}^\gamma \qquad \text{(A.33)} \\
&\hat\Gamma_{\gamma\beta}\wedge\mathring\Gamma_\alpha{}^\gamma + \mathring\Gamma_{\gamma\beta}\wedge\hat\Gamma_\alpha{}^\gamma = \mathring\Gamma_{\gamma\alpha}\wedge\hat\Gamma_\beta{}^\gamma + \hat\Gamma_{\gamma\alpha}\wedge\mathring\Gamma_\beta{}^\gamma = \hat\Gamma_{\gamma(\beta}\wedge\mathring\Gamma_{\alpha)}{}^\gamma + \mathring\Gamma_{\gamma(\beta}\wedge\hat\Gamma_{\alpha)}{}^\gamma
\end{aligned}
$$

Appendix B

The variation of the general lagrangian

We consider a general gauge theory with lagrangian $\mathcal{L} = \mathcal{L}(\Upsilon, d\Upsilon)$, where Υ denotes a collection of arbitrary forms. If Ψ is a matter field and A a gauge potential then, e.g., we would have $\Upsilon = (\Psi, A)$ and we could separate the lagrangian as $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M$, where $\mathcal{L}_G = \mathcal{L}_G(A, dA)$. Minimal coupling means $\mathcal{L}_M = \mathcal{L}_M(\Psi, d\Psi, A)$, whereas in general we might assume $\mathcal{L}_M = \mathcal{L}_M(\Psi, d\Psi, A, dA)$ (see [17] page 67).

[B.1] The *variation of the lagrangian* $\mathcal{L} = \mathcal{L}(\Upsilon, d\Upsilon)$ as we replace $\Upsilon \to \Upsilon + \delta\Upsilon$ reads

$$\delta\mathcal{L} \equiv \mathcal{L}(\Upsilon + \delta\Upsilon, d(\Upsilon + \delta\Upsilon)) - \mathcal{L}(\Upsilon, d\Upsilon) = \delta\Upsilon \wedge \frac{\partial\mathcal{L}}{\partial\Upsilon} + \delta(d\Upsilon) \wedge \frac{\partial\mathcal{L}}{\partial(d\Upsilon)} \,. \tag{B.1}$$

Making this replacement also *in* the exterior derivative $d\Upsilon$ yields $d\Upsilon \to d(\Upsilon + \delta\Upsilon) = d\Upsilon + d(\delta\Upsilon)$ from which we may read off that the exterior derivative varies as $\delta(d\Upsilon) = d(\delta\Upsilon)$, i.e. d and δ commute:

$$[d, \delta] = 0 \,. \tag{B.2}$$

Also, we require the Leibniz rule

$$\delta(\psi_1 \wedge \psi_2) = \delta\psi_1 \wedge \psi_2 + \psi_1 \wedge \delta\psi_2 \tag{B.3}$$

for two arbitrary forms ψ_1 and ψ_2. We rewrite the variation rule (B.1) and apply partial integrations:

$$\delta\mathcal{L} = \delta\Upsilon \wedge \frac{\partial\mathcal{L}}{\partial\Upsilon} + d(\delta\Upsilon) \wedge \frac{\partial\mathcal{L}}{\partial(d\Upsilon)} \tag{B.4}$$

$$= \delta\Upsilon \wedge \frac{\partial\mathcal{L}}{\partial\Upsilon} + d\left[\delta\Upsilon \wedge \frac{\partial\mathcal{L}}{\partial(d\Upsilon)}\right] - (-)^{\mathrm{rank}\Upsilon}\delta\Upsilon \wedge d\frac{\partial\mathcal{L}}{\partial(d\Upsilon)} , \tag{B.5}$$

$$= \delta\Upsilon \wedge \frac{\delta\mathcal{L}}{\delta\Upsilon} + d\left[\delta\Upsilon \wedge \frac{\partial\mathcal{L}}{\partial(d\Upsilon)}\right] . \tag{B.6}$$

Here, we introduced the variational derivative

$$\frac{\delta\mathcal{L}}{\delta\Upsilon} = \frac{\partial\mathcal{L}}{\partial\Upsilon} - (-)^{\mathrm{rank}\Upsilon} d\frac{\partial\mathcal{L}}{\partial(d\Upsilon)} . \tag{B.7}$$

[*B.2*] If we consider the variation $\delta\int_M \mathcal{L}$ of the action with the constraint that $\delta\mathcal{L}$ vanishes at ∂M we may drop the exact part of (B.6). The law $0 = \delta\int_M \mathcal{L}$ then yields the *field equations*. For an interacting system $\mathcal{L} = \mathcal{L}_G(A, dA) + \mathcal{L}_{\mathrm{mat}}(\Psi, d\Psi, A)$, $\mathrm{rank}A = 1$, e.g., the two field equations reads

$$0 = \frac{\delta\mathcal{L}}{\delta A} = \frac{\partial\mathcal{L}_G}{\partial A} + \frac{\partial\mathcal{L}_{\mathrm{mat}}}{\partial A} + d\frac{\partial\mathcal{L}_G}{\partial(dA)} , \tag{B.8}$$

$$0 = \frac{\delta\mathcal{L}}{\delta\Psi} = \frac{\partial\mathcal{L}_{\mathrm{mat}}}{\partial\Psi} - (-)^{\mathrm{rank}\Upsilon} d\frac{\partial\mathcal{L}_{\mathrm{mat}}}{\partial(d\Psi)} . \tag{B.9}$$

Since we only used the Leibniz rule for the exterior derivative all these equations also hold if we start from a lagrangian $\mathcal{L} = \mathcal{L}(\Upsilon, D\Upsilon)$ and replace all derivatives by the covariant.

[*B.3*] Eq (B.6) may in general be split into an exact and a non-exact part. If the variation δ is a local symmetry transformation, the exact part gives an explicit definition of the respective currents and the non-exact part gives the respective *Noether identity*. If, e.g. , $\Upsilon = (A, \Psi)$ and we make a infinitesimal gauge transformation $\Lambda = \mathrm{id} + \omega$, $\omega \in \mathcal{G}$, $\delta A = d\omega$, $\delta\Psi = -\omega\cdot\Psi$, then (B.6) becomes (with $J := \frac{\delta\mathcal{L}}{\delta A}$)

$$\begin{aligned}
\delta\mathcal{L} &= \delta A \wedge \frac{\delta\mathcal{L}}{\delta A} + \delta\Psi \wedge \frac{\delta\mathcal{L}}{\delta\Psi} + d\left[\delta A \wedge \frac{\partial\mathcal{L}}{\partial(dA)} + \delta\Psi \wedge \frac{\partial\mathcal{L}}{\partial(d\Psi)}\right] \\
&= d\omega \dot{\wedge} J - \omega\cdot\Psi \wedge \frac{\delta\mathcal{L}}{\delta\Psi} + d\left[d\omega \dot{\wedge} \frac{\partial\mathcal{L}}{\partial(dA)} - \omega\cdot\Psi \wedge \frac{\partial\mathcal{L}}{\partial(d\Psi)}\right] \\
&= -\omega\cdot dJ - \omega\cdot\Psi \wedge \frac{\delta\mathcal{L}}{\delta\Psi} + d\left[\omega\cdot J - \omega\cdot d\frac{\partial\mathcal{L}}{\partial(dA)} - \omega\cdot\Psi \wedge \frac{\partial\mathcal{L}}{\partial(d\Psi)}\right] .
\end{aligned} \tag{B.10}$$

If we impose $\delta\mathcal{L} = 0$ and extracting the exact part we find

$$J := \frac{\delta\mathcal{L}}{\delta A} = d\frac{\partial\mathcal{L}}{\partial(dA)} + \Psi \wedge \frac{\partial\mathcal{L}}{\partial(d\Psi)} , \tag{B.11}$$

whereas the non-exact part gives the Noether identity

$$dJ = -\Psi \wedge \frac{\delta \mathcal{L}}{\delta \Psi} \stackrel{*}{=} 0. \tag{B.12}$$

The last relation $\stackrel{*}{=}$ hold weekly, i.e. only if the matter field equation (B.9) is satisfied. Eqs (B.11) and (B.12) are equivalent to eqs (5.2.18) and (5.2.16, 5.2.17) in [17].

[*B.4*] Following the previous ideas (and section 5.2.1 in [17]), we display the variation of $\mathcal{L}$ along some vector field ξ, i.e. $\delta\mathcal{L} = \pounds_\xi \mathcal{L}$, $\delta\Upsilon = \pounds_\xi \Upsilon$ with the *covariant* Lie derivative $\pounds_\xi = \xi\rfloor D + D\rfloor\xi$. However, (B.6) does not correctly split into exact and non-exact parts for this variation and we have to restart from (B.4):

$$\begin{aligned}
\delta_\xi \mathcal{L} \equiv \pounds_\xi \mathcal{L} &= D(\xi\rfloor\mathcal{L}) \\
&= (\pounds_\xi \Upsilon)\wedge \frac{\partial \mathcal{L}}{\partial \Upsilon} + (\pounds_\xi D\Upsilon)\wedge \frac{\partial \mathcal{L}}{\partial (D\Upsilon)} && \text{(B.13)} \\
&= (\xi\rfloor D\Upsilon)\wedge \frac{\partial \mathcal{L}}{\partial \Upsilon} + D(\xi\rfloor\Upsilon)\wedge \frac{\partial \mathcal{L}}{\partial \Upsilon} + (\xi\rfloor D(D\Upsilon))\wedge \frac{\partial \mathcal{L}}{\partial (D\Upsilon)} + D(\xi\rfloor D\Upsilon)\wedge \frac{\partial \mathcal{L}}{\partial (D\Upsilon)} && \text{(B.14)} \\
&= (\xi\rfloor D\Upsilon)\wedge \frac{\partial \mathcal{L}}{\partial \Upsilon} + D\left[(\xi\rfloor\Upsilon)\wedge \frac{\partial \mathcal{L}}{\partial \Upsilon} + (\xi\rfloor D\Upsilon)\wedge \frac{\partial \mathcal{L}}{\partial (D\Upsilon)}\right] \\
&\quad - (-)^{\mathrm{rank}\Upsilon - 1}(\xi\rfloor\Upsilon)\wedge D\frac{\partial \mathcal{L}}{\partial \Upsilon} - (-)^{\mathrm{rank}\Upsilon}(\xi\rfloor D\Upsilon)\wedge D\frac{\partial \mathcal{L}}{\partial (D\Upsilon)}\,. && \text{(B.15)}
\end{aligned}$$

Extracting now exact and non-exact parts yields

$$\xi\rfloor\mathcal{L} = (\xi\rfloor\Upsilon)\wedge \frac{\partial \mathcal{L}}{\partial \Upsilon} + (\xi\rfloor D\Upsilon)\wedge \frac{\partial \mathcal{L}}{\partial (D\Upsilon)}\,, \tag{B.16}$$

$$\begin{aligned}
0 &= (\xi\rfloor D\Upsilon)\wedge \frac{\partial \mathcal{L}}{\partial \Upsilon} + (-)^{\mathrm{rank}\Upsilon}(\xi\rfloor\Upsilon)\wedge D\frac{\partial \mathcal{L}}{\partial \Upsilon} - (-)^{\mathrm{rank}\Upsilon}(\xi\rfloor D\Upsilon)\wedge D\frac{\partial \mathcal{L}}{\partial (D\Upsilon)} \\
&= (\xi\rfloor D\Upsilon)\wedge \frac{\delta \mathcal{L}}{\delta \Upsilon} + (-)^{\mathrm{rank}\Upsilon}(\xi\rfloor\Upsilon)\wedge D\frac{\partial \mathcal{L}}{\partial \Upsilon}\,. && \text{(B.17)}
\end{aligned}$$

An important application of these equations is the case of a translational gauge model, $\Upsilon = (\Psi, \vartheta^\alpha)$, when we displace along $\xi = e_\alpha$. Eq (B.16) becomes

$$e_\alpha\rfloor\mathcal{L} = (e_\alpha\rfloor\Psi)\wedge \frac{\partial \mathcal{L}}{\partial \Psi} + (e_\alpha\rfloor D\Psi)\wedge \frac{\partial \mathcal{L}}{\partial (D\Psi)} + \frac{\partial \mathcal{L}}{\partial \vartheta^\alpha} + (e_\alpha\rfloor D\vartheta^\beta)\wedge \frac{\partial \mathcal{L}}{\partial (D\vartheta^\beta)}\,, \tag{B.18}$$

and thereby gives the explicite form of the canonical energy-momentum current $\Sigma_\alpha := \frac{\delta\mathcal{L}}{\delta\vartheta^\alpha}$:

$$\begin{aligned}
\Sigma_\alpha &= \frac{\partial \mathcal{L}}{\partial \vartheta^\alpha} + D\frac{\partial \mathcal{L}}{\partial D\vartheta^\alpha} \\
&= e_\alpha\rfloor\mathcal{L} - (e_\alpha\rfloor\Psi)\wedge \frac{\partial \mathcal{L}}{\partial \Psi} - (e_\alpha\rfloor D\Psi)\wedge \frac{\partial \mathcal{L}}{\partial (D\Psi)} - (e_\alpha\rfloor D\vartheta^\beta)\wedge \frac{\partial \mathcal{L}}{\partial (D\vartheta^\beta)} + D\frac{\partial \mathcal{L}}{\partial (D\vartheta^\alpha)}\,. && \text{(B.19)}
\end{aligned}$$

Eq (B.17) produces the (translational) Noether identity

$$0 = (e_\alpha \rfloor D\vartheta^\beta) \wedge \frac{\delta \mathcal{L}}{\delta \vartheta^\beta} + (e_\alpha \rfloor D\Psi) \wedge \frac{\delta \mathcal{L}}{\delta \Psi} - D\frac{\partial \mathcal{L}}{\partial \vartheta^\alpha} + (-)^{\mathrm{rank}\Psi}(\epsilon_\alpha \rfloor \Psi) \wedge D\frac{\partial \mathcal{L}}{\partial \Psi} \,,$$

$$D\Sigma_\alpha = (e_\alpha \rfloor D\vartheta^\beta) \wedge \Sigma_\beta + (e_\alpha \rfloor D\Psi) \wedge \frac{\delta \mathcal{L}}{\delta \Psi} + (-)^{\mathrm{rank}\Psi}(\epsilon_\alpha \rfloor \Psi) \wedge D\frac{\delta \mathcal{L}}{\delta \Psi} \,, \tag{B.20}$$

$$D\Sigma_\alpha \stackrel{*}{=} (e_\alpha \rfloor D\vartheta^\beta) \wedge \Sigma_\beta \,. \tag{B.21}$$

In the second line we used that $D\frac{\partial \mathcal{L}}{\partial \Upsilon} = D\frac{\delta \mathcal{L}}{\delta \Upsilon}$. The third line holds *weekly*, i.e. only when the matter field equation $0 = \frac{\delta \mathcal{L}}{\delta \Psi}$ is satisfied. Eqs (B.19) and (B.20) are equivalent to eqs (5.2.9) and (5.2.10) in [17].

Appendix C

Brief mathematical reference

Most of the following is an extract from [5], [4], and [3].

C.1 Groups

[*C.1*] A *group* G is a set of elements with an associative multiplication $\circ$ defined on it, such that it includes an identity and an inverse of each element. The maximal set Z of elements commuting with all G is called *center*. The set $G \circ a$ for $a \in G$ is called G-*orbit* of a. A *subgroup* H of G is a subset of G and a group with respect to $\circ$. Thus it contains the identity, its inverse H^{-1}, and is closed. The action of G on itself given by

$$\mathrm{Conj}: \; G \to \mathrm{End}(G): \; a \to \left[\mathrm{Conj}_a : \; b \to a \circ b \circ a^{-1}\right] , \tag{C.1}$$

is called *conjugation*. An *invariant subgroup* H of G is closed with respect to conjugation, i.e. $G \circ H \circ G^{-1} \subset H$. If H is an invariant subgroup of G, the set of G-orbits of all $h \in H$, i.e. G/H, is a group. A mapping h from one group G to another G' preserving the structure, i.e. the multiplication law, is called *homomorphism*. The set $\{g \in G : h(g) = id \in G'\}$ is called *kernel* of h. Endomorphisms, isomorphisms, and automorphisms are defined by interpreting the syllables *endo*∼'on itself', *iso*∼'isomorphic', and *auto*∼'isomorphically on itself'.

[*C.2*] A *representation* $\cdot$ of a group G on a vector space V is a map

$$\cdot : \; G \to \mathrm{End(V)}: \; g \to \left[v \to g \cdot v\right] . \tag{C.2}$$

The representation is *faithful* if $\cdot$ is an isomorphism. For a representation of G on V, a subspace $W \subset V$ is an *ideal* of V if $G \cdot W \subset W$. A representation is *irreducible* if V and $\{\}$ are the only ideals of V that exist.

[*C.3*] A *Lie group* G is a group and differential manifold such that the multiplication $\circ$ is also differential. For each Lie group G there exists a set I of locally isomorphic Lie groups. This set contains exactly one simply connected Lie group $\tilde{G}$ called *covering group* of G. All other groups in I are homomorphic images of $\tilde{G}$ such that the kernel Z is discrete and central. Hence all groups in I are isomorphic to $\tilde{G}/Z$ (for respective Z).

[*C.4*] The *Lie algebra* $\mathcal{G}$ of a Lie group G is $T_{\rm id}G$ with the multiplication $[\ ,\ ]:\ X, Y \to [X, Y] = -[Y, X]$. A representation $\cdot$ of G on V induces a representation $\cdot$ of $\mathcal{G}$ on V by

$$X \cdot v = \frac{d}{dt}[\exp(tX) \cdot v]_{t=0} \qquad \text{for all } v \in V\ . \tag{C.3}$$

Consistently, the concatenation $\circ$ is defined by

$$(X \circ Y) \cdot v = X \cdot (Y \cdot v) \qquad \text{for all } v \in V\ , \tag{C.4}$$

or without a representation

$$X \circ Y = \frac{d}{ds}\frac{d}{dt}\big[\exp(sX) \circ \exp(tY)\big]_{s=0,t=0}\ . \tag{C.5}$$

We find the Baker-Campbell-Hausdorff formula

$$\exp(X) \circ \exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y] + ..)\ . \tag{C.6}$$

Introducing a basis λ_a in $\mathcal{G}$ we define the structure constants $f_{ab}{}^c$ by

$$[\lambda_a, \lambda_b] = f_{ab}{}^c\, \lambda_c\ . \tag{C.7}$$

[*C.5*] The *adjoint representation* of the Lie group G on its algebra $\mathcal{G}$ is the derivative of the conjugation at the identity:

$$\mathrm{Ad}:\ G \to \mathrm{End}(\mathcal{G}):\ a \to\ \mathrm{Ad}_a = d(\mathrm{Conj}_a)(\mathrm{id})\ . \tag{C.8}$$

It follows that $\exp(\epsilon \mathrm{Ad}_a(Y)) = a \circ \exp(\epsilon Y) \circ a^{-1}$ for $\epsilon \to 0$. The adjoint representation of the Lie algebra $\mathcal{G}$ on itself is

$$\mathrm{ad}:\ \mathcal{G} \to \mathrm{End}(\mathcal{G}):\ X \to \big[\mathrm{ad}_X:\ Y \to [X, Y]\big]\ . \tag{C.9}$$

Hence, ad_X corresponds to the matrix $(\mathrm{ad}_X)_b{}^c = X^a f_{ab}{}^c$. The *Cartan metric* k is defined by

$$\begin{aligned} &k(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \qquad \text{or} \\ &k_{ab} = f_{ac}{}^d\, f_{bd}{}^c\ . \end{aligned} \tag{C.10}$$

C.2 Fibre bundles

[*C.6*] A *fibre bundle* (E, π, M, F, G) is a manifold E with *base manifold* M and *typical fibre* (manifold) F such that there exists a surjectiv *projection* $\pi : E \to M$ such that $\pi^{-1}(x)$ is isomorphic to F for any $x \in M$. We also require that, for any open neighborhood $U \subset M$, there exists a diffeomorphic *local trivialization* $\phi_U : \pi^{-1}(U) \to U \times F$. Further, for each pair ϕ_U, ϕ_V of local trivializations with intersectiong U and V there must exist a smooth, G-valued *transition function* $t : U \cap V \to G$ such that $\phi_U^{-1}(x, f) = \phi_V^{-1}(x, t(x) \cdot f)$ for any $x \in U \cap V$ and $f \in F$. Thus, we need a representation $\cdot$ of the *structure group* G on F.

[*C.7*] A *local section* $s_U : U \to E$ is a smooth map that satisfies $\pi \circ s = \mathrm{id}_M$.

[*C.8*] A *principle bundle* (E, π, M, G) is a fibre bundle of which the typical fibre is also its structure group.

[*C.9*] Given a principle bundle (E, π, M, G) one can construct the so-called *associated fibre bundle* as follows: Let F be a manifold on which the action of G is defined. After defining the action $(e, f) \to (g \cdot e, g^{-1} \cdot f)$ of G on $E \times F$, the associated fibre bundle is defined as $(E \times F)/G$ with base manifold M, typical fibre F, and structure group G. The projection is trivially given by considering $E \times F \to M : (e, f) \mapsto \pi(e)$ on the quotient space.

[*C.10*] A *connection* A on a principle bundle with structure group G is a $\mathcal{G}$-valued 1-form which satisfies

$$\tilde{X} \rfloor A = X \qquad \text{for } X \in \mathcal{G} \text{ and } \tilde{X} = \frac{d}{dt} \exp(tX) \in TP \;, \tag{C.11}$$

$$g \cdot A = \mathrm{Ad}_{g^{-1}}(A) \;. \tag{C.12}$$

[*C.11*] Given a local section s_U one associates a 1-form $\bar{\psi}$ on the base space M with a 1-form ψ on E via the pullback $\bar{\psi} = s^*\psi$. Given a $\mathcal{G}$-valued 1-form $\bar{A}$ on M there exists a unique connection

$$A = g^{-1}(\pi^* \bar{A})g + g^{-1}(dg) \;, \tag{C.13}$$

where g is the local '*choice of gauge*' defined by the local trivialization of the section $\phi_U(\sigma(x)) = (x, g)$. (See [4] section 10.1.)

C.3 Topology

C.3.1 Homotopy, homology, and cohomology

[*C.12*] *Homotopy.*
Let N and M be manifolds. M connected and $\partial M = 0$. $I := [0,1]$. Two maps $f_1, f_2 : N \to M$ are homotopic if there exists a continuous map $\tau : N \times I \to M$ with $\tau(N,0) = f_1$ and $\tau(N,1) := f_2$. τ is then called homotopy starting at f_1.

[*C.13*] *Homotopy group.*
The class $\{f\}$ of homotopic $f : S^n \to M$ is called the n-dimensional homotopy group $\pi_n(M)$. The manifold M is called k-connected if $\forall i \leq k : \pi_i(M) = 0$

[*C.14*] *Euler characteristic.*
Let D^a be the a-dimensional open ball. The Euler characteristic $\chi(X)$ of some topological space X is calculated by polyhedronisation. In three dimensions the Euler characteristic is

$\chi(X) =$ (number of vertices) - (number of edges) + (number of faces).

The Gauss-Bonnet theorem [*C.31*] gives another way of calculating the Euler characteristic.

[*C.15*] *Simplices.*
The r-simplex is defined as subspace of $\mathbb{R}^r$:

$$\sigma_r = \{x|\ x_i > 0\ ,\ \sum_{i=1..r} x_i < 1\}\ . \tag{C.14}$$

It follows that the number of q-faces of an r-simplex is $\binom{r+1}{q+1}$. A simplex is denoted by $r+1$ points $\langle p_0 \dots p_r \rangle$. By introducing an orientation of the simplexes one writes e.g. $(p_0 p_1) = -(p_1 p_0)$ for a 2-simplex. A set of 'nicely fitting' simplices that are 'glued together' is called a simplicial complex. Let K be a simplicial complex. $|K|$ is the according polyhedron of K. If there exists a homeomorphism $f : |K| \to X$ then X is called trianguable. A simplicial complex K can be written as set of complexes $\{\sigma_\alpha\}$.

[*C.16*] *Chain group.*
The r-chain group $C_r(K)$ of a simplicial complex K is the free Abelian group generated by the oriented r-simplexes of K. If $r > \dim K$ the group is defined to be 0. An element of $C_r(K)$ is called an r-chain.

[*C.17*] *Boundary operator.*

The boundary of an r-simplex is defined as

$$\partial_r \sigma_r = \sum_{i=0}^{r} (-1)^i (p_0 \dots \hat{p}_i \dots p_r) \tag{C.15}$$

The exact sequence $C_n \to C_{n-1} \to \dots \to C_0 \to 0$ by applying $\partial_n \dots \partial_0$ is called chain complex.

[*C.18*] *Homology group.*
$c \in C_r(K)$ is called r-cycle if $\partial_r c$=0. $c \in C_r(K)$ is called r-boundary if there exists a $d \in C_{r+1}$ such that $c = \partial_{r+1} d$. The set of r-cycles $Z_r(K)$ and the set of r-boundaries $B_r(K)$ are subgroups of $C_r(K)$. The rth homology group of K is defined by $H_r(K) = Z_r(K)/B_r(K)$

Homology groups are topological invariants, i.e. if two spaces are homeomorph they have the same homology group.

[*C.19*] *de Rham Cohomology group.*
Let M be a differentiable manifold. We denote the set of closed r-forms with $Z^r(M)$ and the set of exact r-forms with $B^r(M)$. The quotient $H^r(M) = Z^r(M)/B^r(M)$ denotes the rth de Rham cohomology group of M.

[*C.20*] *de Rham theorem.*
The map

$$\Lambda : \; H_r(M) \times H^r(M) \to \mathbb{R} \, , \quad ([c],[w]) \mapsto \int_c w \tag{C.16}$$

is bilinear and non-degenerate. Thus $H^r(M)$ is the dual vector space of $H_r(M)$.

C.3.2 The Chern-Weil theorem

[*C.21*] *Invariant, symmetric r-forms over $\mathcal{G}$.*
Let $\mathcal{G}$ be the algebra of the group G. Let $\tilde{P} : \times_r \mathcal{G} \to \mathbb{C}$ be a totally symmetric, r-linear form. $\tilde{P}$ is *invariant* if

$$\forall g \in G, A_i \in \mathcal{G} \; : \; \tilde{P}(\mathrm{Ad}_g A_1, .., \mathrm{Ad}_g A_r) = \tilde{P}(A_1, .., A_r) \; . \tag{C.17}$$

[*C.22*] *Invariant polynomials of degree r over $\mathcal{G}$.*
Let $A \in \mathcal{G}$ and $\tilde{P}$ be invariant. $P(A) := \tilde{P}(A, .., A) : \mathcal{G} \to \mathbb{C}$ is an *invariant polynomial of degree r.*

[*C.23*] *Invariant polynomials over* $\Lambda^k(\mathcal{G})$.
Let $\eta_i \in \Lambda^{p_i}(M)$. A r-linear form $\tilde{P} : \times_r \Lambda^k(\mathcal{G}) \to \Lambda^k(\mathbb{C})$ is defined by

$$\tilde{P}(A_1\eta_1, .., A_r\eta_r) := \tilde{P}(A_1, .., A_r)\, \eta_1 \wedge .. \wedge \eta_r \; . \tag{C.18}$$

Analogously we define the polynomial $P : \Lambda^p(\mathcal{G}) \to \Lambda^p(\mathbb{C})$.

[*C.24*] *Chern-Weil theorem I.*
Let $F \in \Lambda^i(M, \mathcal{G})$ and P be an invariant polynomial of degree r. It follows that $dP(F) = 0$. Hence the equivalent class $[P(F)] \in H^{ri}(M)$.

[*C.25*] *Chern-Weil theorem II.*
Let F and F' be curvatures corresponding to two different connections on the same bundle E. Then $P(F) - P(F')$ is exact. Hence $[P(F)]$ depends only on the bundle itself, i.e. its topology, but is independent of F. We call $[P(F)]$ characteristic class. Any polynomial $P(F) = 1 + P_1(F) + P_2(F) + ..$ is a direct sum of the jth terms $P_j(F) \in \Lambda^{2j}(E, \mathbb{C})$ of even degree. We call $[P_j(F)]$ the jth characteristic class.

[*C.26*] *Chern-Simons forms.*
Let $[P_j(F)]$ be a jth characteristic class. Form the Chern-Weil theorem it follows that the $2j$-form $P_j(F)$ is closed. Hence locally there exists a $2j - 1$ form Q with $P_j(F) = dQ(\Gamma, F)$ that in general also depends on the potential Γ of F. We call Q Chern-Simons form of $P_j(F)$. Let $U \subset M$ be the domain of Q, then

$$\int_U P_j(F) = \int_{\partial U} Q(\Gamma, F) \; . \tag{C.19}$$

That means that $[Q(\Gamma, F)] \in H^{2j-1}(\partial U)$ specifies the topology of ∂U.

[*C.27*] *Chern class.*
The polynomial $\det(1 + A)$ is invariant. Hence, if F is the curvature on the bundle E, the polynomial $c(F) := \det(1 + F)$ defines the Chern class $[c(F)]$ and the jth Chern classes $[c_j(F)]$. The jth Chern terms $c_j(F)$ explicitly read

$$\begin{aligned} &c_0(F) = 1 \; , \quad c_1(F) = \mathrm{tr}F \; , \quad c_2(F) = \frac{1}{2}\left[\mathrm{tr}F \wedge \mathrm{tr}F - \mathrm{tr}(F \wedge F)\right] \; , \quad ... \; , \\ &c_k(F) = \det F \end{aligned} \tag{C.20}$$

[*C.28*] *Chern character class.*
The polynomial $\mathrm{tr}\exp(A)$ is invariant. Hence, if F is the curvature on the bundle E, the polynomial $ch(F) := \mathrm{tr}\exp(F)$ defines the Chern character class $[ch(F)]$ and the jth Chern

character classes $[ch_j(F)]$. The jth Chern character terms $ch_j(F)$ explicitly read

$$ch_j(F) = \frac{1}{j!}\mathrm{tr}F^j \tag{C.21}$$

[*C.29*] *Pontrjagin class.*
The Pontrjagin class $[p(F)]$ is defined by the invariant polynomial $p(F) = \det(1 + F/2\pi)$ and is the real analogue to the Chern class. Since this polynomial is even in F, the jth Pontrjagin terms $p_j(F)$ are of order $2j$ in F and hence $p_j(F) \in H^{4j}(M)$. The terms $p_j(F)$ have the same form as $(-1)^i c_{2j}(F)$.

[*C.30*] *Euler class.*
Let M be a n-dimensional Riemannian manifold with curvature R. The Euler term $e(R)$ is that n-form that (formally) satisfies $e(F) \wedge e(R) = p_{n/2}(R)$. If n is odd we define $e(F) = 0$.

[*C.31*] *Gauss-Bonnet theorem.*
If $\chi(M)$ is the Euler characteristic and $e(R)$ the Euler term of a manifold with curvature R we have

$$\int_M e(R) = \chi(M) \ . \tag{C.22}$$

Appendix D

Computer algebra

The following Reduce files implement standard solution of Einstein gravity and the Poincaré gauge theory of gravity. We tried to collect standard routines that reflect the theoretical structure and that are necessary for any implementation of a solution in one library called `magtools`. These routines include the definition of the Christoffel symbol, contortion, torsion, and curvature in our conventions. Further routines implement the irreducible decompositions of the torsion and curvature given in [17], the definition of the general field momenta (excitations) and the lagrangian, and the field equations FIRST and SECOND. Also the teleparallel and Einsteinian cases are considered. The routines concerning the spatial integrations and the charge definitions are merely tools for chapter 2. The author hopes that this library is continuously updated and generalized by himself and other physicists in this area of research. The subsequent files `kerrnut.exi`, `mccrea.exi`, and `baekler.exi` are good examples of how to implement solutions with this library very quickly. Their output, that one can find at the end of each file, literally represents eqs (2.16, 2.21, 2.26, 2.29). The last file `proca.exi` implements the coupled Einstein-Proca system as described in chapter 3 and exports the field equations for further manipulations with Maple. The Maple file performing the integration and presentation of the Einstein-Proca solution is far too extensive to display it here. But, as all other files, one may find it at
`http://www.thp.uni-koeln.de/~mt/work/1999dipl/` .

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Diplomarbeit selbstständig verfasst und alle benutzten Quellen und Hilfsmittel vollständig angegeben habe. Teilveröffentlichungen:

> M. Toussaint: A gauge theoretical view of the charge concept in Einstein gravity. Gen. Rel. Grav., accepted for publication (1999/2000), Los Alamos e-Print Archive `gr-qc/9907024`.

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