Mathematics for Intelligent Systems
Lecture 2 Homework
(Linear Algebra II: Projections and The Fundamental Structure of Linear Transforms)

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1 Problem 1

During the lecture we have seen the formula for projecting a vector v onto a vector p. Let us call this transform P:

$$P = \frac{p\langle p, \cdot\rangle}{p^2} \,. \tag{1}$$

We will now learn how to derive this formula, and extend it so that we can express more complex projections.

Any projection P decomposes input vectors v into a parallel component $v_{//}$ and an orthogonal component $v_\perp$. These two components satisfy the following properties:

$$v = v_{//} + v_\perp \tag{2}$$

$$Pv = v_{//} = \alpha p \tag{3}$$

These equalities should be geometrically intuitive, make sure you understand them well.

a. We have all the tools necessary to prove 1. Start with $\langle pv\rangle$, and proceed by substituting the decomposition of v. [Hint: what is $Pv_\perp$?]

b. We now derive the projection P onto a *set* of orthogonal vectors $\{p_i\}_i$. To do this, we need to update one of the previous equations:

$$Pv = v_{//} = \sum_k \alpha_k p_k \,. \tag{4}$$

How does this projection relate to the previous? [Hint: Remember $\{p_i\}_i$ are *orthogonal*!]

c. What is the projection transform $P_\perp$ which projects onto the orthogonal component $v_\perp$? [Hint: you can use the *identity* transform I]

2 Problem 2

In problem 1 we saw a case where non-orthonormal bases made the problem somewhat more difficult. This problem explores the geometry of non-orthonormal bases in more detail.

1. Let $\mathcal{B}_{\mathcal{V}} = \{a_i\}_{i=1}^n$ be an arbitrary basis in vector space $\mathcal{V}$ with generic inner product $\langle u, v\rangle$. Write an expression for the inner product in terms of the coordinates in basis $\mathcal{B}_{\mathcal{V}}$. Specifically, if $u = \sum_i \alpha_i a_i$ and $v = \sum_i \beta_i a_i$, what is the inner product $\langle u, v\rangle$ in terms of the vectors of coordinates $\alpha = (\alpha_1; \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$? [Hint: we've seen what the expression is when the basis is *orthonormal*, which isn't the case here]

2. We can always interpret the space of coefficients to be our abstract vector space (defined by coordinate-wise addition and scalar multiplication). But the inner product between two such vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ may take an arbitrary form $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{y}$, where $\boldsymbol{A}$ is a positive definite matrix. If that's our inner product, what can we conclude about our chosen basis representation with respect to this inner product? What linear transform of our space would "stretch" it so that this inner product is correctly represented as a traditional "dot" product between transformed vectors? [Hint: this is related with the singular value decomposition of A]