Mathematics for Intelligent Systems
Lecture 3 Homework
(Review of Vector Spaces, Bases, Matrix Representations of Linear Transforms)

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1 Problem 1: Vector Spaces

The following are basic questions about vector spaces which should be really clear to all of you. These will not count towards the number of exercises you need to be able to take the exam, but you should still make sure that you understand each case. We will only review them briefly in class, mostly we'll review the ones in which you are most uncertain.

1. Is the set of real numbers $\mathbb{R}$ a vector space?
2. Is the set of complex numbers $\mathbb{C}$ a vector space?
3. Is the set of integers $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ a vector space?
4. Is the set of all derivable functions a vector space?
5. Is the set of all functions f which are continuous everywhere except at $x = 17$ a vector space?
6. Is the set of all functions f which are continuous everywhere except at one point a vector space?
7. Is the set of all convex functions a vector space?
8. Is the set of all *non-even* functions a vector space?
9. Is the set of all finitely integrable functions f a vector space? (A function is finitely integrable if $\int_{-\infty}^{\infty} f(x)dx < \infty$)
10. Is the set of all functions which integrate to 1 ($\int_{-\infty}^{\infty} f(x)dx = 1$) a vector space?
11. Is the set of polynomials $\alpha_0 + \alpha_1 x + \dots \alpha_n x^n$ where $\alpha_i \in \mathbb{R}$ a vector space?
12. Is the set of polynomials $\alpha_0 + \alpha_1 x + \dots \alpha_n x^n$ where $\alpha_i \in \mathbb{Z}$ a vector space?
13. Is the set of polynomials $x^{\alpha_0} + x^{\alpha_1} + \dots x^{\alpha_n}$ where $\alpha_i \in \mathbb{R}$ a vector space?

14. Is the set of functions $\alpha_0 \cos(\pi x) + \alpha_1 \sin(2\pi x)$ where $\alpha_i \in \mathbb{R}$ a vector space?

15. Is the set of functions $\alpha_0 \exp(\frac{1}{2}x^2) + \alpha_1 \exp(x)$ where $\alpha_i \in \mathbb{R}$ a vector space?

16. Is the set of functions $\exp(\frac{1}{2}\alpha_0 x^2)+\exp(\alpha_1 x)$ where $\alpha_i \in \mathbb{R}$ a vector space?

17. Is the set of all colors a vector space?

2 Problem 2: Vector spans

These exercises will count, and we'll see them individually.

1. Which space do vectors $1, 1 - x^2, x^2 - x^4, x^4 - x^6, \ldots$ span?

2. Which space do vectors $1 + i, 1 - i$ span?

3. Describe the space spanned by quaternions $1, i, 1 + i, j + k$. Does it span the whole quaternion space?

4. f is a linear function on the space of quaternions. Given that $f(1-i+j-k) = a$, $f(i - j + k) = b$, $f(j - k) = c$, $f(k) = d$ what are $f(1), f(i), f(j)$ and $f(1 + i + j + k)$?

3 Problem 3: Coordinates, change of basis and transformations

The set of n-degree polynomials $\sum_i \alpha_i x^i$ is a vector space and $v = 1 + 2x + 3x^2$ is a vector in that space. Let us denote two bases as $B = \{1, x, x^2, \ldots\}$ and $B' = \{1, 1 + x, 1 + x + x^2, \ldots\}$.

1. What are $[v]_B$ and $[v]_{B'}$, the coordinates of v in these two bases?

2. Assume function f maps a polynomial onto a its own degree, e.g. $f(2x + 3x^4) = 4$. If f a linear function?

3. What are $f([v]_B)$ and $f([v]_{B'})$ and why?

4. What matrix M allows you to convert between coordinates $[v]_B$ and $[v]_{B'}$, i.e. $[v]_{B'} = M[v]_B$? Which matrix M' does the same in the opposite direction, i.e. $[v]_B = M'[v]_{B'}$? What is the relationship between M and M'?

5. What does the difference between coordinates $[v]_B - [v]_{B'}$ represent?

6. Assume y and z are two vectors in this space, and that the coordinates w.r.t to basis B are $[y]_B = (1, 2, 1, 2, \ldots)$ and $[z]_B = (2, 1, 2, 1, \ldots)$. What *vector* does the coordinate difference $[y]_B - [z]_B$ correspond to? What if the above coordinates were w.r.t basis B'?

7. Take the linear transformation T which maps $1 \mapsto 1$, $x \mapsto 1 + x$, $x^2 \mapsto 1+x+x^2$, etc... Find the matrix M which performs this operation in basis B, i.e. $[Tv]_B = M[v]_B$. Do the same for basis B', i.e. $[Tv]_{B'} = M'[v]_{B'}$.

8. Continuing from the previous question, how do you obtain the matrix which simultaneusly performs a change of basis (from basis B to basis B' *and* the transformation T? i.e. find matrix $\tilde{M}$ such that $[Tv]_{B'} = \tilde{M}[v]_B$.

9. The identity transformation I is one which maps $v \mapsto v$. What is the matrix M which performs the identity transformation in all possible iterations of input and output basis? i.e. find the matrices M such that

   - $[Iv]_B = M[v]_B$
   - $[Iv]_{B'} = M[v]_{B'}$
   - $[Iv]_{B'} = M[v]_B$
   - $[Iv]_B = M[v]_{B'}$

   and what is the relationship of these matrices with the change of basis matrices?

4 Problem 4: Inner product and Orthogonality

1. Show that $f(x, y) = 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2$ is an inner product on $\mathbb{R}^2$.

2. In the space of functions with the inner product $\langle fg \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$, what is the projection of $\sin(x)$ onto $\sin(2x)$? (Graphical argument is ok)

3. Given a vector space V, a basis $\{e_i\}$ and the *metric tensor* (the matrix of basis vector inner products) $g_{ij} = \langle e_i e_j \rangle$, how does one compute $\langle vw \rangle$ for any $v, w \in V$? In which case does this become equivalent to the dot product $v \cdot w = \sum_i v_i w_i$?

4. What property does a matrix M has to satisfy in order to be a valid metric tensor, i.e. such that $x^\intercal My$ is a valid inner product?