Mathematics for Intelligent Systems Lecture 8 Homework
(Optimization)

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1 Equality Constraint Penalties and Augmented Lagrangian

Take a squared penalty approach to solving a constrained optimization problem

$$\min_x \; f(x) + \mu \sum_{i=1}^{m} h_i(x)^2 \tag{1}$$

The Augmented Lagrangian method adds yet another penalty term

$$\min_x \; f(x) + \mu \sum_{i=1}^{m} h_i(x)^2 + \sum_{i=1}^{m} \lambda_i h_i(x) \tag{2}$$

Assume that if we minimize (1) we end up at a solution $\bar{x}$ for which each $h_i(\bar{x})$ is reasonable small, but not exactly zero. Prove, in the context of the Augmented Lagrangian method, that setting $\lambda_i = 2\mu h_i(\bar{x})$ will, if we assume that the gradients $\nabla f(x)$ and $\nabla h(x)$ are (locally) constant, ensure that the minimum of (2) fulfills the constraints $h(x) = 0$.

Tip: Think intuitive. Think about how the gradient that arises from the penalty in (1) is now generated via the λ_i.

2 Lagrangian and dual function

(Taken roughly from 'Convex Optimization', Ex. 5.1)

A simple example. Consider the optimization problem

$$\min x^2 + 1 \text{ s.t.} (x-2)(x-4) \le 0$$

with variable $x \in \mathbb{R}$.

a) Derive the optimal solution x^* and the optimal value $p^* = f(x^*)$ by hand.

b) Write down the Lagrangian $L(x, \lambda)$. Plot (using gnuplot or so) $L(x, \lambda)$ over x for various values of $\lambda \ge 0$. Verify the lower bound property $\min_x L(x, \lambda) \le p^*$, where p^* is the optimum value of the primal problem.

c) Derive the dual function $l(\lambda) = \min_x L(x, \lambda)$ and plot it (for $\lambda \geq 0$). Derive the dual optimal solution $\lambda^* = \arg\max_\lambda l(\lambda)$. Is $\max_\lambda l(\lambda) = p^*$ (strong duality)?