An Introduction to Riemannian Geometry

Jim Mainprice, PhD, *Humans to Robots Research (HRM) Group*
Outline

1 - Why geometry matters
   • Feature maps
   • Dimensionality reduction

2 - Differential geometry
   • Manifolds
   • Differentiable Maps
   • Diffeomorphisms
   • Tangent spaces

3 - Riemannian geometry
   • Riemannian metric
   • Calculus on the sphere
   • Pullback metric
   • Induced metric
   • Nash embedding theorem
   • Geodesics
   • Harmonic maps

4 - Example in motion optimization
Why Geometry Matters

*Shortest path* on the globe is not the same as *shortest path* on the map

[Slide Courtesy: Frank C. Park RSS17]
2D Maps of the Globe

[Slide Courtesy: Frank C. Park RSS17]
Nonlinear Feature Maps

- Feature maps are nonlinear transformation of the data space
- The classification problem is linear in $\mathcal{H}$ but not in $X$
Dimensionality Reduction

- Data lies on an embedded non-linear manifold
- **Manifold learning**: Find an embedding to Euclidean space that minimally distorts the metric
  - Can be used for feature extraction
  - *Embedding* is **nonlinear**, *Metric* in feature space is **linear**
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What is a Manifold

- A manifold $M$ is a topological space
  - Set of points with neighborhood for each point
- Each point of $M$ has a neighborhood homeomorphic to Euclidean space
- A **coordinate chart** is a pair $(U, \varphi)$

$$\varphi : U \rightarrow \hat{U} \subseteq \mathbb{R}^n$$

Example: 4 charts of the circle
What is an *smooth*-Manifold

- An **atlas** $A$ is a collection of coordinate charts that cover the manifold
  - said to be smooth if there exists *transition maps* between its charts
- A **smooth-manifold** $(M, A)$ is a topological manifold with a smooth atlas

Transition map between two coordinate charts

![Diagram](image_url)
Differentiable Maps

- A map $f$ is said to be **differentiable** if its **first order derivative** exists for all points of the domain $U$.

- $f$ is said of class $C(r)$ if it has continuous partial derivative up to order $r$.

- When $r = \infty$, $f$ is said to be **smooth**.

$$f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\frac{\partial^r f(x)}{\partial x^r}, \quad \forall x \in U$$
Jacobian and Hessian

- **Jacobian** matrix is the matrix of first order derivative of $f$

- **Hessian** is the matrix of second order derivatives of $f$

- **Differential** $d$ of $f$ at $x$ is the map solution of the first order linearization of $f$

\[
\mathbf{J}_f(x) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{pmatrix}
\]

\[
\mathbf{H}_f(x) = \begin{pmatrix}
\frac{\partial^2 f_1}{\partial x_1^2} & \cdots & \frac{\partial^2 f_1}{\partial x_n^2} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 f_m}{\partial x_1^2} & \cdots & \frac{\partial^2 f_m}{\partial x_n^2}
\end{pmatrix}
\]

\[
d_x f : \mathbb{R}^n \to \mathbb{R}^m
\]

\[
s.t. \\
f(x + \delta) = f(x) + d_x f(\delta) + o(\delta)
\]
What is a Diffeomorphism

• A differentiable map \( f : M \rightarrow N \) between two manifolds \( M \) and \( N \) is called a diffeomorphism if
  • it is a \textit{bijection}
  • its \textit{inverse} \( f^{-1} : N \rightarrow M \) is \textit{differentiable}

• \textbf{Inverse Function Theorem} states that a differentiable map is invertible \textit{iif} it’s Jacobian determinant does not vanish on the domain
smooth-Manifolds

Transition maps are **diffeomorphisms**, hence their Jacobians are invertible because they map euclidean spaces of the same dimension.
Examples of Manifolds with Global Charts

- Euclidean Space \( \mathbb{R}^m \)

- Implicit smooth surfaces defined by
  \[
  f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^m
  \]

- Space of n-dimensional Gaussian distributions
  - (Euclidean point, Positive matrix)
  \[
  \varphi = (\mu, \Sigma) \in \mathbb{R}^n \times P(n)
  \]
**Example: 1D Gaussian Distributions**

- The space of 1D Gaussian distribution is parametrized by mean and standard deviation.

- Can also be parameterized by mean and variance.

- Smooth reparametrization i.e., **change of coordinates**

\[ \psi \circ \varphi^{-1}(x) = (\mu, \sqrt{\sigma^2}) \]
Example: The Sphere

\[ S^2 = \{(x, y, z) \in R^3 \mid x^2 + y^2 + z^2 = 1\} \]

- In general, manifolds require **multiple coordinate charts**
- Here it needs two charts that can be obtained by stereographic projection

\[ \varphi_N(p) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right), \text{ with } p \in S^2 \]
Submanifolds of Euclidean Space

• A submanifold of Euclidean space can be defined through the kernel of a nonlinear smooth map \( f \)

• Dimension of the manifold is \( m - n \)

• For example, the unit sphere \( S^n \) can be defined this way
  • \( n = 1, m = \dim(E) \)

\[ f : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n, \text{ with } n \leq m \]

\[ M = f^{-1}(0) \]

Sphere as a submanifold of Euclidean space

\[ f(x) = 1 - \|x\|^2 \]
Product Manifolds

- If $M$ and $N$ are manifolds, so is $M \times N$
- The resulting **chart is the product of the charts** on both manifolds

Cylinder $= S^1 \times U \subset \mathbb{R}$

Torus $= S^1 \times S^1$
What Are Tangent Vectors

- Tangent vectors indicate a **direction** on the manifold.
- To specify a tangent vector we can specify a curve on the manifold and take the **derivative of the curve** at the point \( p \) on \( M \).

\[
\gamma_n : [-1, 1] \mapsto M
\]
What is the Tangent Space

- Suppose two differentiable curves are given

- Equivalent at \( p \) iff the derivative of their pushfoward through a local-coordinate chart \( \varphi \) coincide at 0

\[
\varphi \circ \gamma_1, \varphi \circ \gamma_2 : (-1, 1) \rightarrow \mathbb{R}^n
\]

- Any such curves leads to an equivalence class denoted:

\[
v = \dot{\gamma}(0)
\]

- The **tangent space** of \( M \) at \( p \), \( T_p M \), is then defined as the set of all tangent vectors at \( p \); it does not depend on the choice of coordinate chart
Tangent Space Basis

• Consider the set of all smooth function on M: \( C^\infty(M) \)

• The tangent space can also be defined as the set of all derivations:

\[
D : C^\infty(M) \to \mathbb{R}
\]

• A basis for the tangent space can be defined for a given coordinate chart in the following way:

\[
\forall i \in \{1, \ldots, n\}, \forall f \in C^\infty(M) : \\
\left( \frac{\partial}{\partial x^i} \right)_p (f) \overset{df}{=} (\partial_i (f \circ \varphi^{-1}))(\varphi(p))
\]
Differential of Smooth Maps

- Consider a smooth map $f$. A differential (or pushforward) is a function that maps the tangent spaces of $M$ and $N$.

- It corresponds to the **Jacobian of $f$ in local coordinates**.

\[ T_{f_p}N \]
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Riemannian metric

- A Riemannian metric is a an inner product $g(u, v)$ defined over the tangent space varying smoothly over $M$

- In coordinates, it corresponds to a smooth family of positive-definite matrices

- A smooth manifold with a Riemannian metric is a **Riemannian manifold** $(M, g)$
Riemannian of Euclidean Space

- In Euclidean space, the associated metric is constant and equal to the identity

\[ \forall p \in \mathbb{R}^n, \forall u, v \in T_p\mathbb{R}^n, g(u, v) = u^T v \]

\[ A_p = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} = I \]
Calculus on the Sphere

Sphere is defined by

\[ S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \]

Coordinates:

\[ \varphi(x, y, z) = \left( \tan^{-1}\left(\frac{y}{x}\right), \cos^{-1}(z) \right) = (\theta, \phi) \]

A curve on the sphere \( \gamma(t) = (x(t), y(t), z(t)) \) has the following arc length

\[ ds^2 = dx^2 + dy^2 + dz^2 = d\phi^2 + \sin^2\phi \, d\theta^2 \]
Calculus on the Sphere

Calculating lengths and area on the sphere using spherical coordinates:

- Length on the sphere
  \[ C = \int_0^T \sqrt{\dot{\phi}^2 + \dot{\theta}^2 \sin^2 \phi} \, dt \]

- Area on the sphere
  \[ A = \iint_A |\sin(\phi)| \, d\phi \, d\theta \]

[Slide Courtesy: Frank C. Park RSS17]
Pullback Metric and Embeddings

The pullback metric of $g$ through $f$

$$(f^* g^N)(v, w) = g^N(df(v), df(w))$$

In coordinates:

$$f^* g^N = J^T A^N J$$

$f$ defines an embedding of $M$ in some other manifold $N$ if it is an injective homeomorphism,

$J$ is the Jacobian of $f$ at $p$.
Induced Metric on Submanifolds of Euclidean space

- The induced metric is the inner product from Euclidean space restricted to $M$
- We can get it by pullback of the identity through a coordinate-chart

$$A_p = J_f^T J_f = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}$$

$$f(\theta, \phi) = \begin{cases} x = \sin \theta \cos \phi \\ y = \sin \theta \sin \phi \\ z = \cos \theta \end{cases}$$

$$J_f(\theta, \phi) = \begin{pmatrix} \cos \theta \cos \phi & -\sin \theta \sin \phi \\ \cos \theta \sin \phi & \sin \theta \cos \phi \\ -\sin \theta & 0 \end{pmatrix}$$
Nash Embedding Theorems

- Every Riemannian manifold can be isometrically embedded into some Euclidean space
  - Under particular conditions $n \geq m+1$
  - Harder conditions lead to $n \geq m(3m+11)/2$

$$A_p = J_f^T J_f$$
What are Geodesics

• Geodesics correspond to curves for which

\[ \nabla \dot{\gamma} \dot{\gamma} = 0 \]

where \textit{nabla} is an affine connection

intuitively this means that the acceleration is either 0 or orthogonal to the tangent space, it \textbf{generalizes the notion of straight line to curved spaces}

Connections allow to differentiate tangent vectors from different tangent spaces, we will not see them.
Geodesic Equation

- Geodesic locally minimize the length functional

\[ L(\gamma) = \int_0^T \sqrt{g_{\gamma}(t)(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt \]

- The solution to the Euler-Lagrange equation linked to the minimization of this functional leads to the geodesic equation:

\[
\frac{d^2 x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0
\]

In coordinates (the Christoffel symbols characterize the metric connection)

With this definition of length every Riemannian Manifold can become a metric space.

The geodesic equation has a unique solution, given an initial position and an initial velocity.
Coordinate Invariant Distortion Measure

- Consider a smooth map $f : M \to N$ between two compact Riemannian manifolds
  - $(M, g)$: local coord. $x = (x_1, \ldots, x_m)$, Riemannian metric $G$
  - $(N, h)$: local coord. $y = (y_1, \ldots, y_n)$, Riemannian metric $H$
  - Isometry
    - Map preserving length, angle, and volume - the ideal case of no distortion
  - if $\dim(M) \leq \dim(N)$, $f$ is an isometry when for all $p$ in $M$

$$J_f^T H(f(p)) J_f(p) = G(p)$$

[Slide Courtesy: Frank C. Park RSS17]
Coordinate Invariant Distortion Measure

- Comparing the pullback metric $J^T H J(p)$ to $G(p)$

- Global distortion measure:

$$\int_M \sigma(\lambda_1, \cdots, \lambda_m) \sqrt{\det G} \, dx^1 \cdots dx^m$$

with lambda roots of $\det(J^T H J - \lambda G) = 0$

[Slide Courtesy: Frank C. Park RSS17]
Harmonic Maps

• Define the functional

\[ E(f) = \int_M tr(J^T H J G^{-1}) \sqrt{\det G} \, dx^1 \cdots dx^m \]

• Variational equation throughout Euler-Lagrange formula, solutions are known as the harmonic maps (ones that locally minimize E)

• Intuitively these f wraps marble (N) with rubber (M), **harmonic maps correspond to elastic equilibria**

[Slide Courtesy: Frank C. Park RSS17]
Example of Harmonic Maps

- **Lines**
  \[ E(f) = \int_0^1 \dot{f}^2 \, dt \]

- **Geodesics**
  \[ E(f) = \int_0^1 \dot{f}^T H \dot{f} \, dt \]

- **Laplace’s Equation**
  \[ \nabla^2 f = 0 \]

- **2R Spherical mechanism**
  \[ E(f) = \pi^2 (\epsilon_1 + 2\epsilon_2) \]
Things to Know about Riemannian Manifolds

• Riemannian metrics can be used to define various properties of a manifold such as
  • angles, lengths of curves, areas (or volumes)
  • curvature
  • gradients of functions
  • divergence of vector fields.

• Every manifold admits a Riemannian metric

• Can be used to model most of modern physics
  • Theory of gravitation
  • Theory of electro-magnetism
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The space around us is fundamentally non-Euclidean.
How can we warp workspace geometry?

Geodesics

\[ x^* = \arg\min_{x \in W(x_1, x_2)} \frac{1}{2} \int \dot{x}^T B(x) \dot{x} \, dt \]

Euclidean workspace metric

\[ B(x) = I \]

Riemannian workspace metric

\[ B(x) = B_{WS}(x) \]

[Mainprice IROS16]
Pullback Metric Definition

\[ \phi_{WS}(x) = \begin{bmatrix} \alpha_1 \phi_1(x) \\ \vdots \\ \alpha_{d+1} \phi_{d+1}(x) \end{bmatrix} \]

**Workspace Geometry map**
- Decomposition in local coordinate systems
- Geodesics well defined

**Euclidean space**
\[ B(x) = I \]

**Riemannian Workspace**
\[ \mathcal{W} = (\mathbb{R}^3, B_{WS}(x) = J_{\phi_{WS}}^T J_{\phi_{WS}}) \]

[RieMO, Ratliff 15]

[Mainprice IROS16]
How does the metric operate on a manipulation problem?

Riemannian Workspace Metric

\[ B_{WS}(x) = J_{\phi WS}^T J_{\phi WS} \]

“A Pullback” metric

\[ A(x) = J_{\phi}^T B(x) J_{\phi} \]

\[ \| \dot{x} \|^2_{B(x)} = \dot{x}^T B(x) \dot{x} = \dot{q}^T (J_{\phi}^T B(x) J_{\phi}) \dot{q} \]

\[ \dot{x} = J_{\phi} \dot{q} \]
Local coordinates around arbitrary objects?

- Spherical coordinate system
- Cylindrical coordinate system

Point cloud

Mesh

[Mainprice IROS16]
Electrical Coordinate Systems

Electric potential at \( \mathbf{p} \)

\[
P(\mathbf{p} \in \mathbb{R}^3) = \int \frac{\rho(\mathbf{x})}{||\mathbf{p} - \mathbf{x}||_2} d\Omega(\mathbf{x})
\]

Charge simulation (i.e., find charges \( e_i \))

\[
\begin{align*}
& e_1 P(T_1, p_1) + \ldots + e_n P(T_n, p_1) = V \\
& \vdots \\
& e_1 P(T_1, p_n) + \ldots + e_n P(T_n, p_n) = V
\end{align*}
\]

System has 3 coordinates

- \( r \) : value of the potential
- \( u \) : 1st coordinate of field line
- \( v \) : 2nd coordinate of field line

[Harmonic parameterization by electrostatics, Wang 13]

[Mainprice IROS16]
Motion Optimization Objective

Finite time horizon cost functional

\[
c(\xi) = \int_0^T \sum_{i=1}^m c_{ti}(q_t, \frac{d}{dt}\phi_i(q_t), \ldots, \frac{d^k}{dt^k}\phi_i(q_t))dt
\]

with discretized trajectory using \(N\) way points

\[
\xi = [q_1 \ldots q_N]^T
\]

Results in a differentiable function network

[Mainprice IROS16]
Augmented Lagrangian Optimizer

\[ \xi = \begin{bmatrix} q_1 & \cdots & q_N \end{bmatrix}^T \]

while not StopCondition() do

\[ \begin{align*}
v & \leftarrow f(\xi_i) ; \\
g\xi & \leftarrow \nabla f(\xi_i) ; \\
\xi_{i+1} & \leftarrow \xi_i - \eta_i B_i^{-1} g\xi_i ;
\end{align*} \]
Voxelized Electrical Potential Field (EPF)

2 cm resolution, Triangle mesh color proportional to the charge value
Fixed Goal Configurations Comparisons

- **STOMP** 0th order
- **CHOMP** 1st order
- "Gauss-Newton" 2nd order

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Goal Sets (Null space) Comparisons

- **STOMP** 0th order
- **CHOMP** 1st order
- "**Gauss-Newton**" 2nd order

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That’s it !!! (links are clickable)


• **Aasa Feragen and François Lauze** (2014). *A Very Brief Introduction to Differential and Riemannian Geometry.*
  *(Presentation)* University of Copenhagen. Faculty of Science.

  *(Keynote)* Robotics Science and System (RSS).

• **Jim Mainprice, Nathan Ratliff, and Stefan Schaal** (2016). *Warping the workspace geometry with electric potentials for motion optimization of manipulation tasks.*
  *(Paper)* IROS.